MATH 538: Techniques in Applied Mathematics

Assignment 3

Due Tuesday, March 21 by 3pm

- hand in to me personally or slip under door of CAB 539

1. **(/15)**
   Use the Poincaré-Lindstedt method to obtain a two-term perturbation approximation to the following initial value problem:

$$y'' + y = \epsilon y(1 - y'^2), \quad y(0) = 1, y'(0) = 0.$$

2. **(/20)**
   Obtain the leading order inner and outer solutions and find the composite (uniform) approximation of the following singular boundary value problem in $y(x)$ for $0 \leq x \leq 1$:

$$\epsilon y'' + 2y' + e^y = 0,$$

   with $y(0) = y(1) = 0$ and $0 < \epsilon \ll 1$.

   Do NOT use the WKB approximation for this problem.

3. **(/20)**

   Find the $O(\epsilon)$ accurate solution of the following differential eigenvalue problem:

$$(1 - x^2)y'' - 2xy' + \lambda y = \epsilon y^3$$

   where $-1 \leq x \leq 1$ and, to leading order bounded solutions are given by

$$y \simeq y_0(x) = Ax, \text{ for } \lambda \simeq \lambda_0 = 2.$$

   Here A is a known, fixed constant.

   (You will need to find the adjoint differential equation, $L^\dagger y^\dagger = 0$, for this problem with non-constant coefficients. You can look this up in texts, but still show explicitly using integration by parts that $\int y^\dagger L y \, dx = \int y L^\dagger y^\dagger \, dx = 0$.)

4. **(/15)**
A bead of mass m can slide on a circular hoop of radius R that rotates about a vertical diameter with constant angular velocity ω (see Figure 1). The position of the bead (in terms of its angle θ to the vertical in radians) is given by Newton's laws:

$$mR\frac{d^2\theta}{dt^2} = mR\omega^2 \cos\theta \sin\theta - mg\sin\theta$$

a) Find the equilibrium (steady state) solutions.
b) Sketch the graph of the steady solutions as a function of $\omega > 0$.
c) Use linearized perturbation analysis to examine the stability of the solutions about the equilibrium solutions.
d) Using your results in c) and b), sketch the bifurcation diagram, indicating the position of the bifurcation point.

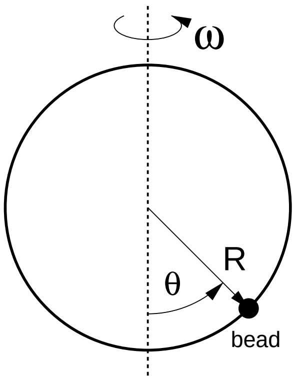

Figure 1: Bead sliding on a rotating hoop (see question 4).

5. **(/20)**
Use the Liouville transformation and the WKB method to find an approximate solution for $y(t)$ to the initial value problem for the strongly damped spring:

$$my'' + ay' + ky = 0, \quad y(0) = 0, \ my'(0) = I.$$

(Generally, the Liouville transformation $u = y\exp[\frac{1}{2}\int^x p(\xi)\,d\xi]$ converts the differential equation $y'' + p(x)y' + q(x)y = 0$ into a Schrödinger equation of the form $u'' + r(x)u = 0$.)

Note: you may compare your solution with exact theory, but do not rely on exact theory to arrive at your solution.

6. **(/10)**
Use WKB theory to find the large eigenvalues of the boundary value problem for $0 < x < 1$:

$$y'' + \lambda\exp(4x)y = 0, \quad y(0) = y(1) = 0.$$

Find the corresponding eigenfunctions.