MATH 538: Techniques in Applied Mathematics

Assignment 2

Due Friday, February 10 by 4pm

- hand in to me personally or slip under door of CAB 539

1. **(/20)**
   A focused laser blast heats an infinitesimally small spherically symmetric volume to a temperature U_0 within a large chunk of plastic. The heat then proceeds to diffuse radially outwards in all directions. The corresponding distribution of temperature of the plastic, $u(\vec{r}, t)$, is given by the solution of the heat equation

   $$\frac{\partial u}{\partial t} = \kappa \nabla^2 u,$$

   in which κ is the heat diffusivity.

   Assuming the structure of the diffusing heated volume remains spherically symmetric for all time, use dimensional analysis to convert the partial differential equation to an ordinary differential equation. Thus find the general solution for u given in terms of well-known analytic functions (not integrals of functions).

2. **(/20)**
   Give two-term expansions in ϵ for the 3 roots of the cubic polynomial

   $$\epsilon^2 x^3 + x^2 + 2x + \epsilon$$

   in which $\epsilon \ll 1$.

3. **(/20)**
   Find the first two terms in the small ϵ expansion for the solution of

   $$\sqrt{2}\sin(x + \pi/4) - 1 - x + x^2/2 = -\epsilon/6$$

   about $x = 0$.

4. **(/20)** For the following system of equations with $\epsilon \ll 1$:

   $$\begin{aligned} (1-\mu)a &= \epsilon(a^2 + b) \\ (2-\mu)b &= \epsilon(a + b), \end{aligned}$$

   find the approximate solution near $\mu = 2$, assuming that $b = B$ is a known, fixed value. Specifically, give the first three non-zero terms of the expansion in ϵ for a, b and μ.

5. **(/20)**

Two buckets of equal cross-sectional area A are connected by a U-tube at the bottom, as illustrated in Figure 1. The bucket on the right is filled initially to a depth h_0 with salt water of density ρ_0 and that on the left is filled to a depth h_1 with fresh water of density ρ_1. So that the fluid does not flow spontaneously from one bucket to another, $h_1 = h_0\rho_0/\rho_1$.

After time $t = 0$ the salt water is extracted at a constant rate Q. Meanwhile fresh water displaces into the salt water bucket and is instantaneously mixed reducing the density of fluid in this bucket.

In terms of the instantaneous salt water density $\rho(t)$, salt water depth $h(t)$ and fresh water depth $\eta(t)$, the coupled nonlinear ordinary differential equations governing the evolution of the system are prescribed respectively by weight balance, volume conservation and mass conservation:

$$\rho_1 A\eta = \rho A h \tag{1}$$

$$\frac{d}{dt}(A\eta + Ah) = -Q \tag{2}$$

$$\frac{d}{dt}(\rho_1 A\eta + \rho A h) = -Q\rho \tag{3}$$

with initial conditions $\rho(0) = \rho_0$, $h(0) = h_0$, $\eta(0) = h_1$

We wish to find $\rho(t)$ for all time until both buckets are empty under the assumption that $\epsilon \equiv (\rho_0 - \rho_1)/\rho_1 \ll 1$.

a) Choose appropriate characteristic scales for ρ, h, η and t. Write the nondimensional form of ρ, $\tilde{\rho}$, in the form $1 + \epsilon r(\tilde{t})$ in which r is a function of order unity. Use these substitutions to convert your formulae into nondimensional form.

a) Manipulate the formulae to derive a single nonlinear ordinary differential equation: Use (1) to eliminate (nondimensional) η from (2) and (3). From the first of these results integrate in time and use the initial conditions to come up with an explicit formula relating (nondimensional) h to r. Use this in the second of your results to determine an equation for $r(\tilde{t})$ alone.

c) Use perturbation theory for differential equations to find the leading non-zero term in the expansion for $r(\tilde{t})$ in terms of ϵ. Convert your answer to dimensional form to arrive at an approximate formula for $\rho(t)$.

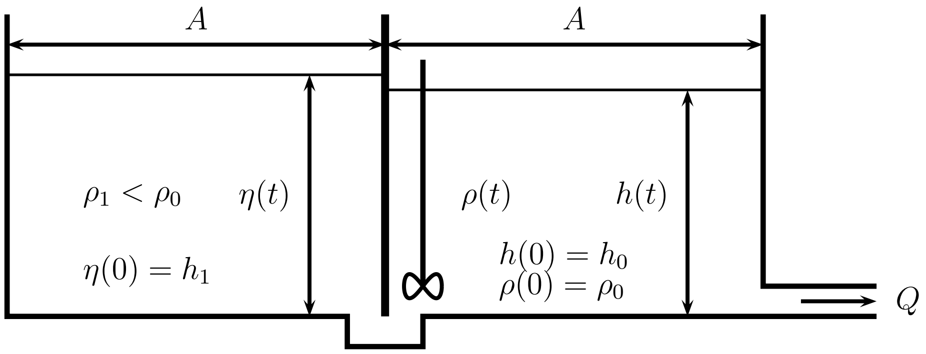

Figure 1: Schematic of mixing problem described in question 6.