The Multispectral Method: Progress and Prospects

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Shell Models of Turbulence: Modes

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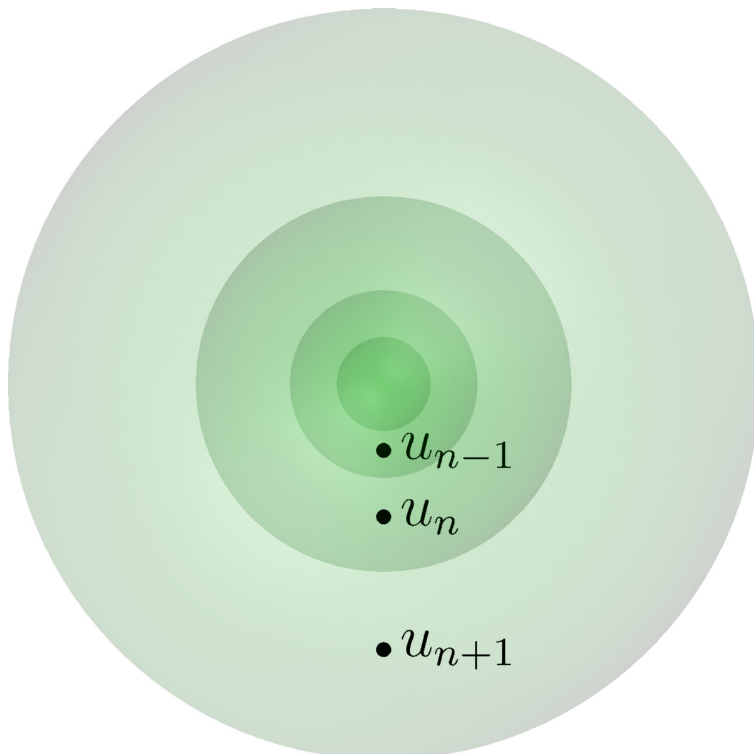
- Collections of modes $\{\boldsymbol{u_k} : k \in [\lambda^n, \lambda^{n+1})\}$ are represented by a single quantity u_n:



Shell Models of Turbulence: Interaction

- The convolution is replaced with a quadratic function of u:

$$\frac{du_n}{dt} = k_n \sum_{p,q} c_{p,q} u_p u_q - \nu k_n^2 u_n.$$

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$$\frac{du_n}{dt} = ik_n \left(a_n u_{n-1}^2 - \lambda a_{n+1} u_n u_{n+1} + b_n u_{n-1} u_n - \lambda b_{n+1} u_{n+1}^2\right)^* - \nu k_n^2 u_n.$$

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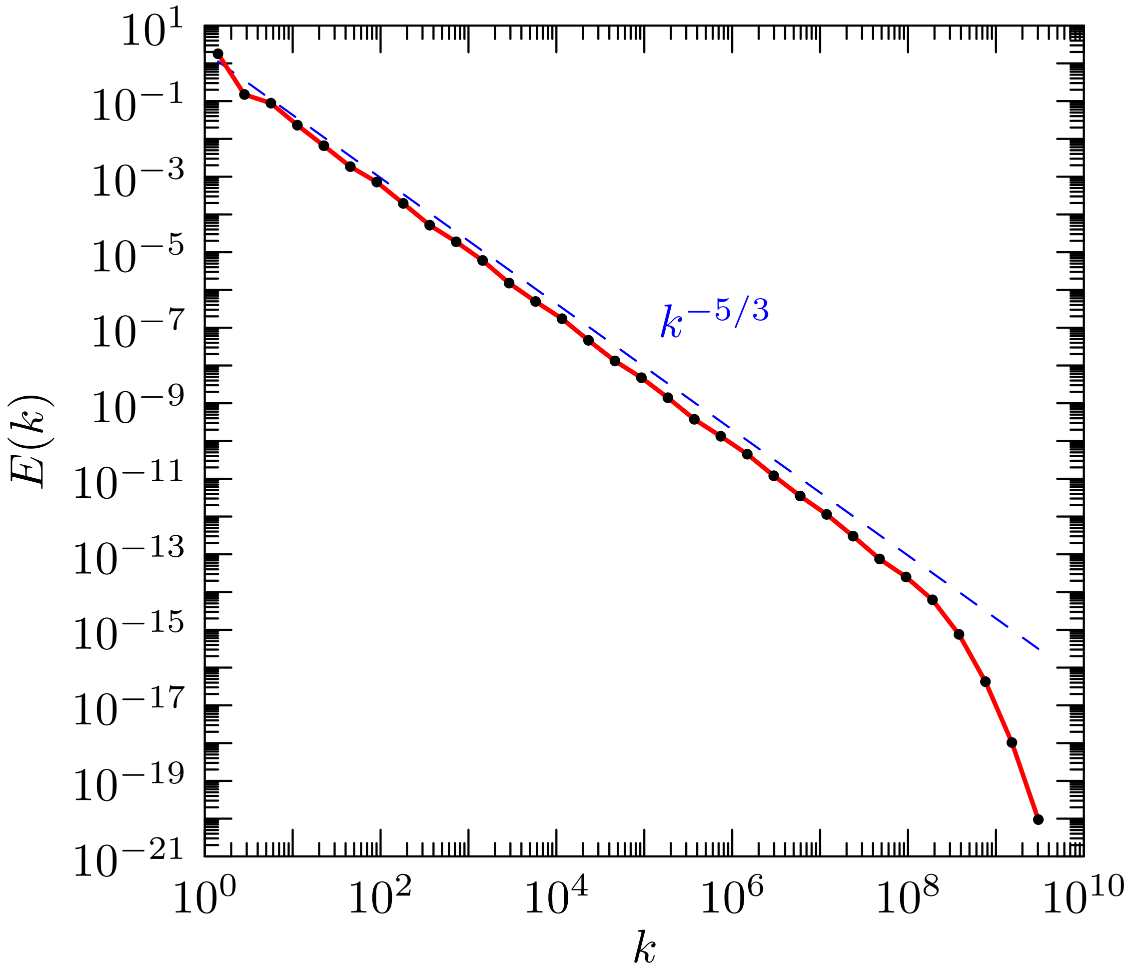
- The GOY [Gledzer 1973, Yamada & Ohkitani 1987] model adds next-nearest-neighbour interactions and conserves the helicity $H = \frac{1}{2} \sum_n (-1)^n k_n |u_n|^2$:

$$\frac{du_n}{dt} = ik_n \left(\alpha u_{n+1} u_{n+2} + \frac{\beta}{\lambda} u_{n-1} u_{n+1} + \frac{\gamma}{\lambda^2} u_{n-1} u_{n-2} \right)^* - \nu k_n^2 u_n.$$

Shell Models: Kolmogorov Scaling

- Simulations reproduce a $k^{-5/3}$ Kolmogorov inertial range:



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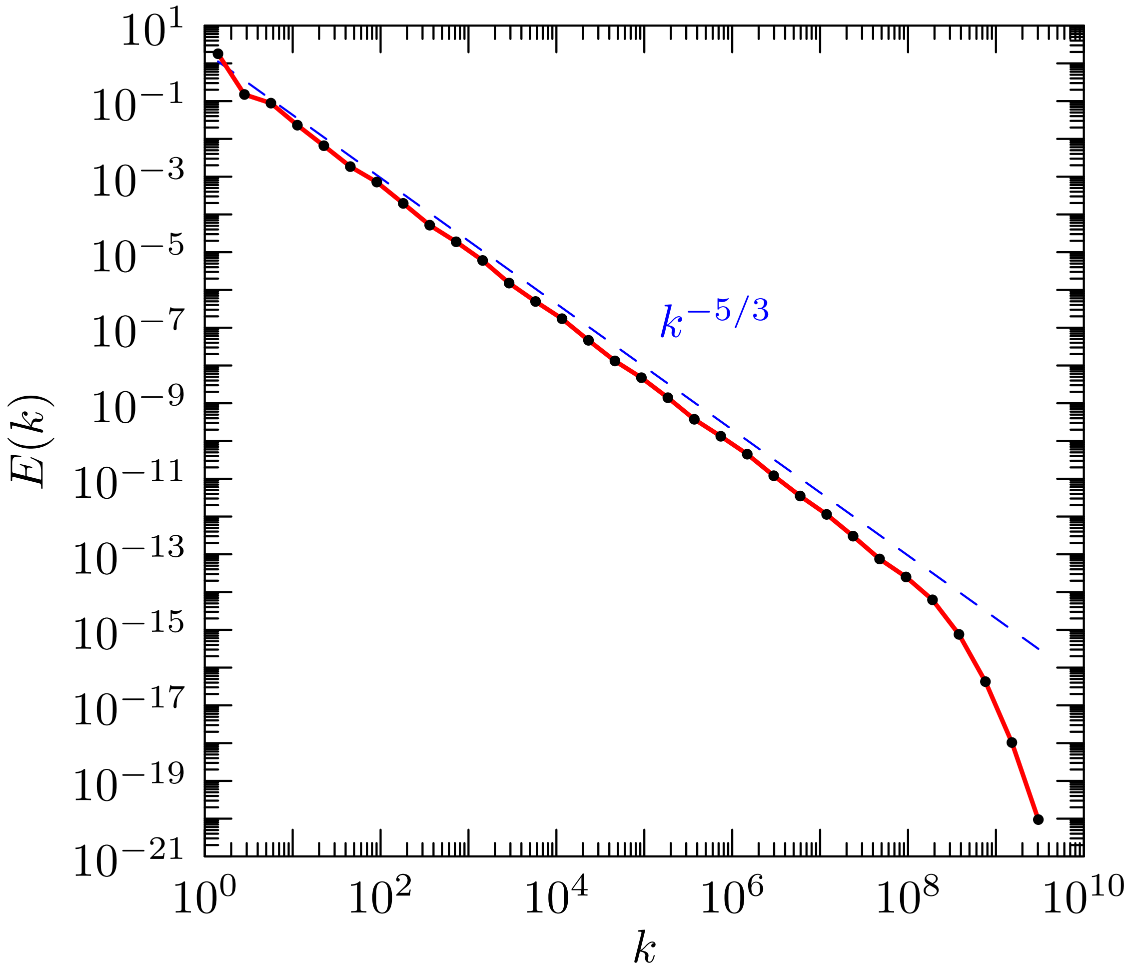
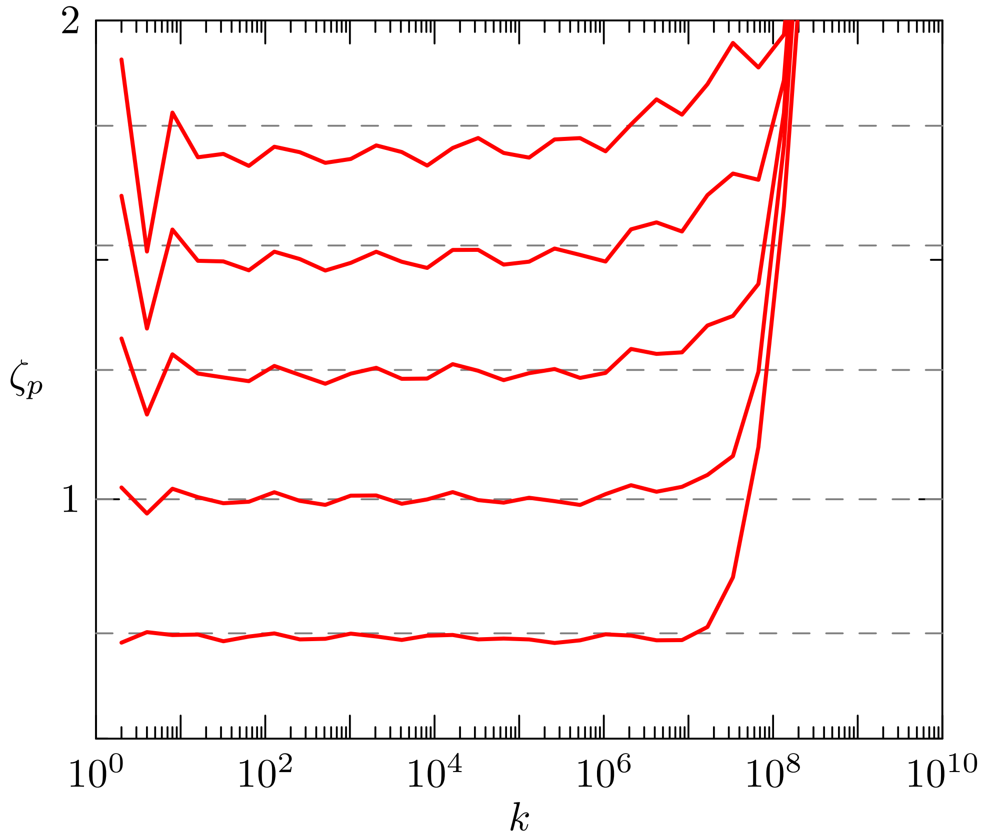


- Shell models are simpler and easier to simulate than the Navier–Stokes equations [Bowman *et al.* 2006].

Shell Models: Intermittency

- They also reproduce statistical properties of Navier–Stokes turbulence: the moments $\langle |u_n|^p \rangle \sim k_n^{-\zeta_p}$



scale very much like experimental structure exponents for 3D turbulence (dashed lines) [Herweijer & van de Water 1995].

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$$u_{n,1} \doteq \frac{u_{2n} + \sigma_n^* u_{2n+1}}{1 + |\sigma_n|^2}, \qquad \sigma_n \doteq \frac{u_{2n+1}}{u_{2n}}.$$

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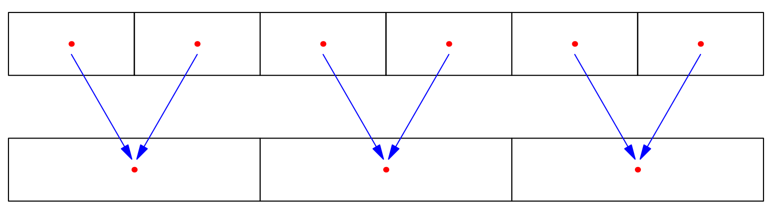
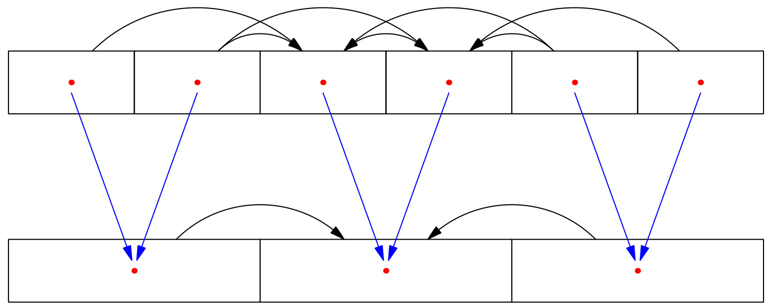
- This reduces the number of active modes by half:



Fixed Point

- Spectral reduction reduces the GOY model to the DN model, which is a fixed point.



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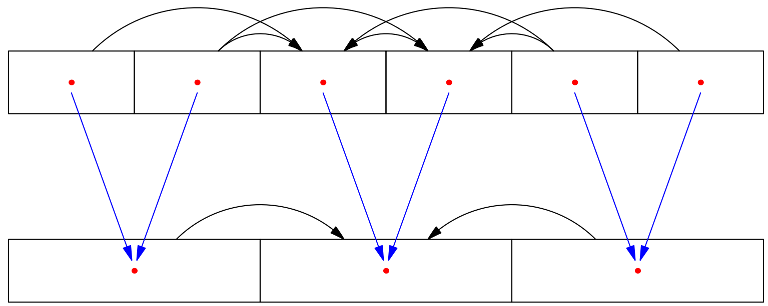

- Further reduction is straightforward:

$$u_{n,\ell+1} \doteq \frac{u_{2n,\ell} + \sigma^*_{n,\ell} u_{2n+1,\ell}}{1 + |\sigma_{n,\ell}|^2}, \quad \sigma_{n,\ell} \doteq \frac{u_{2n+1,\ell}}{u_{2n,\ell}}.$$

$$\text{GOY} \xrightarrow{\text{reduction}} \text{DN} \xrightarrow{\text{reduction}} \text{DN} \xrightarrow{\text{reduction}} \text{DN}$$

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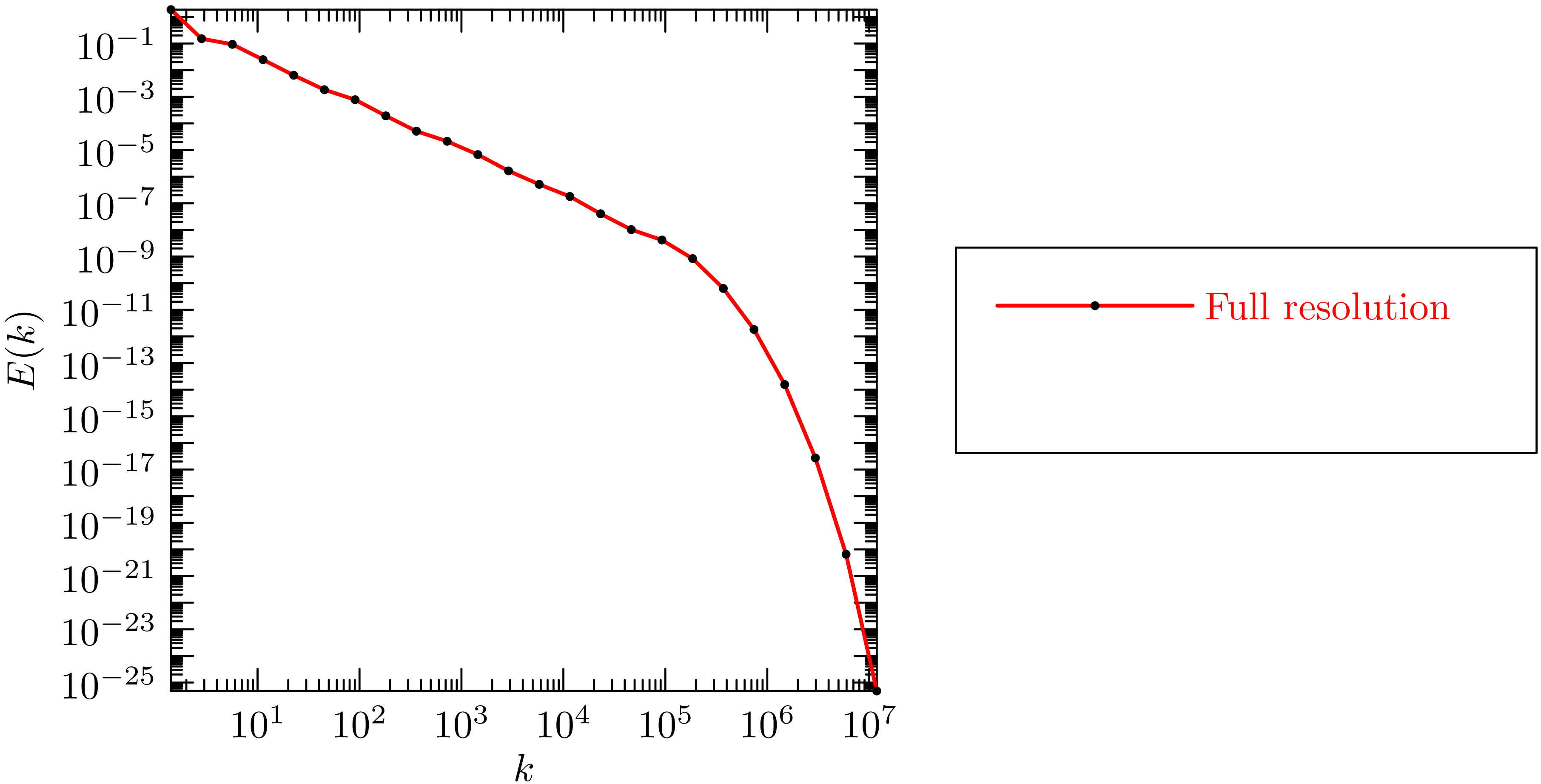
- Binning modifies the viscous term and the interaction coefficients:

$$(\alpha, \beta, \gamma) \to (a, b) \doteq \left(\frac{\gamma}{\lambda^2}, -\frac{\alpha}{\lambda} \right) \to \frac{(a, b)}{2}.$$

Decimation



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$E(k)$
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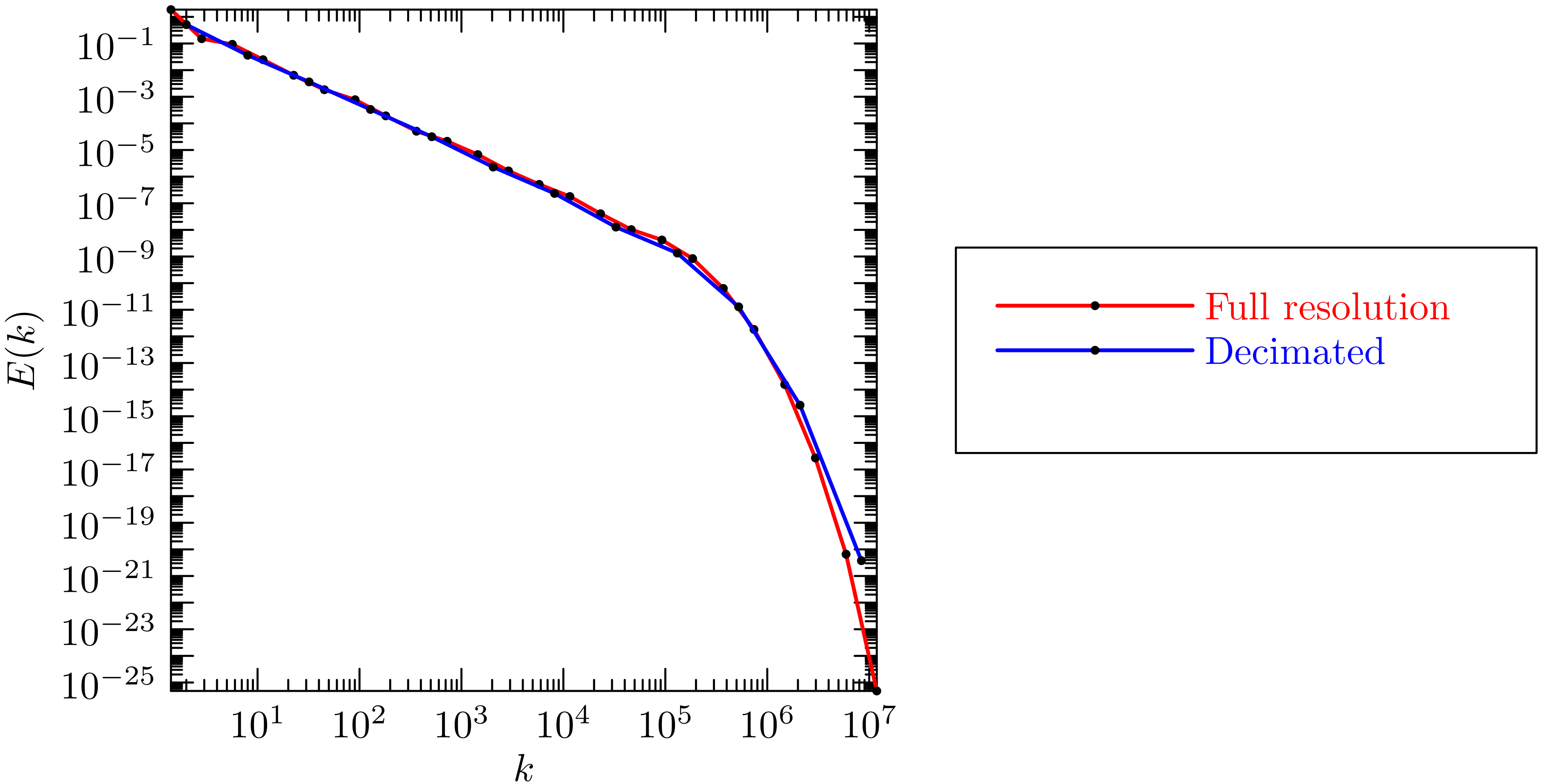
k
Full resolution

Decimation



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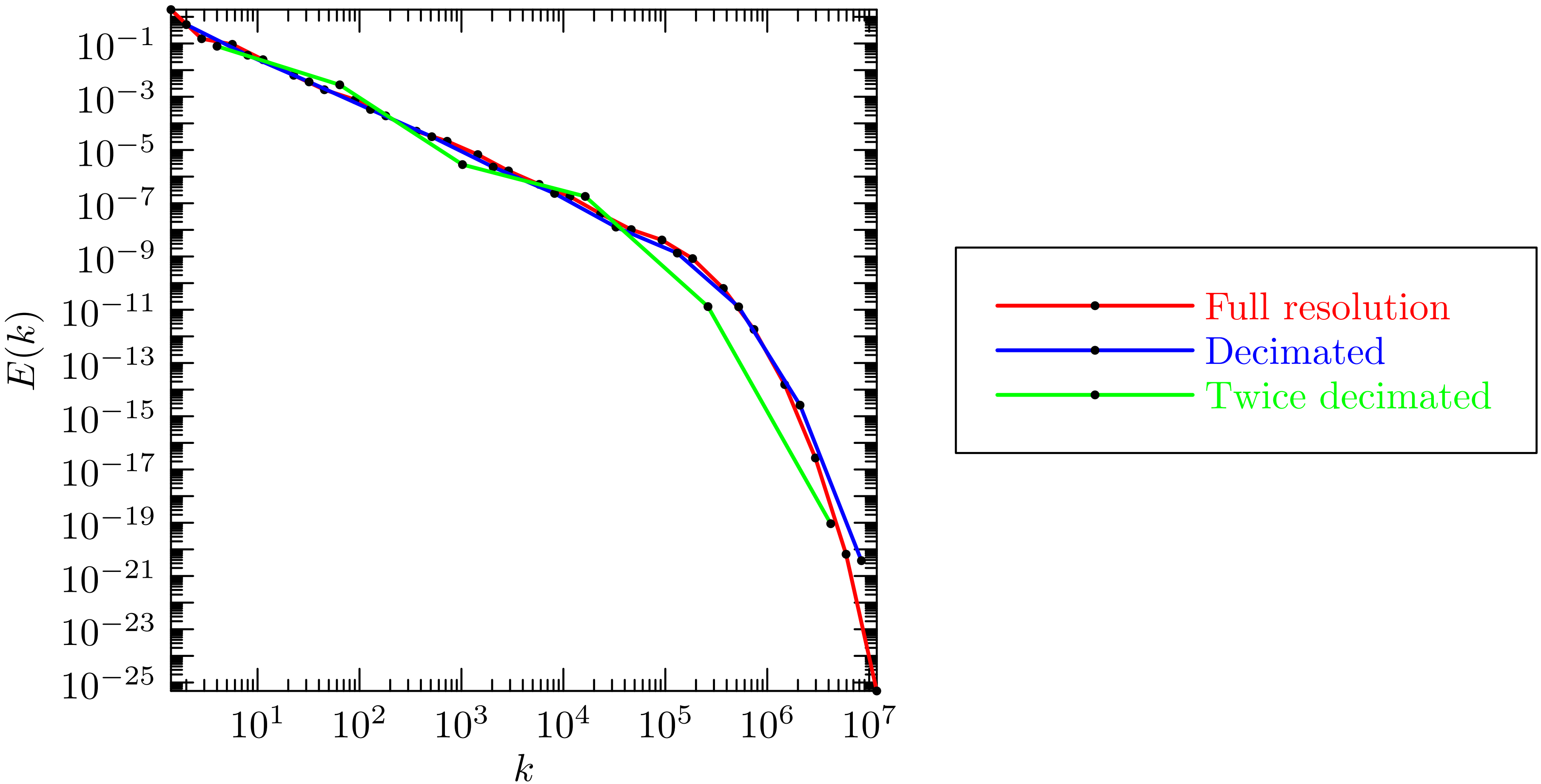

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10^7
Full resolution
Decimated
Twice decimated

Decimation: Interpolation

- Approximating the (unresolved) quantity u_{2n+1} by $\sqrt{u_{n+1,1}u_{n,1}}$ yields

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- A cubic spline can be used for smoother interpolation.

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- Under interpolation, the evolution equation is of the form

$$\frac{du_{n,1}}{dt} = \frac{k_{n,1}}{1+|\sigma_n|^2}\Big[a\left(\sigma_{n-1}u_{n,1}^2 - \lambda^2\sigma_n u_{n,1}u_{n+1,1}\right) \\ + b\left(\sigma_{n-1}u_{n-1,1}u_{n,1} - \lambda^2\sigma_n u_{n+1,1}^2\right)\Big]^* - \nu_{n,1}k_{n,1}^2 u_{n,1}.$$

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and the nonlinear source is divided by $\left(1+|\sigma_n|^2\right)$.

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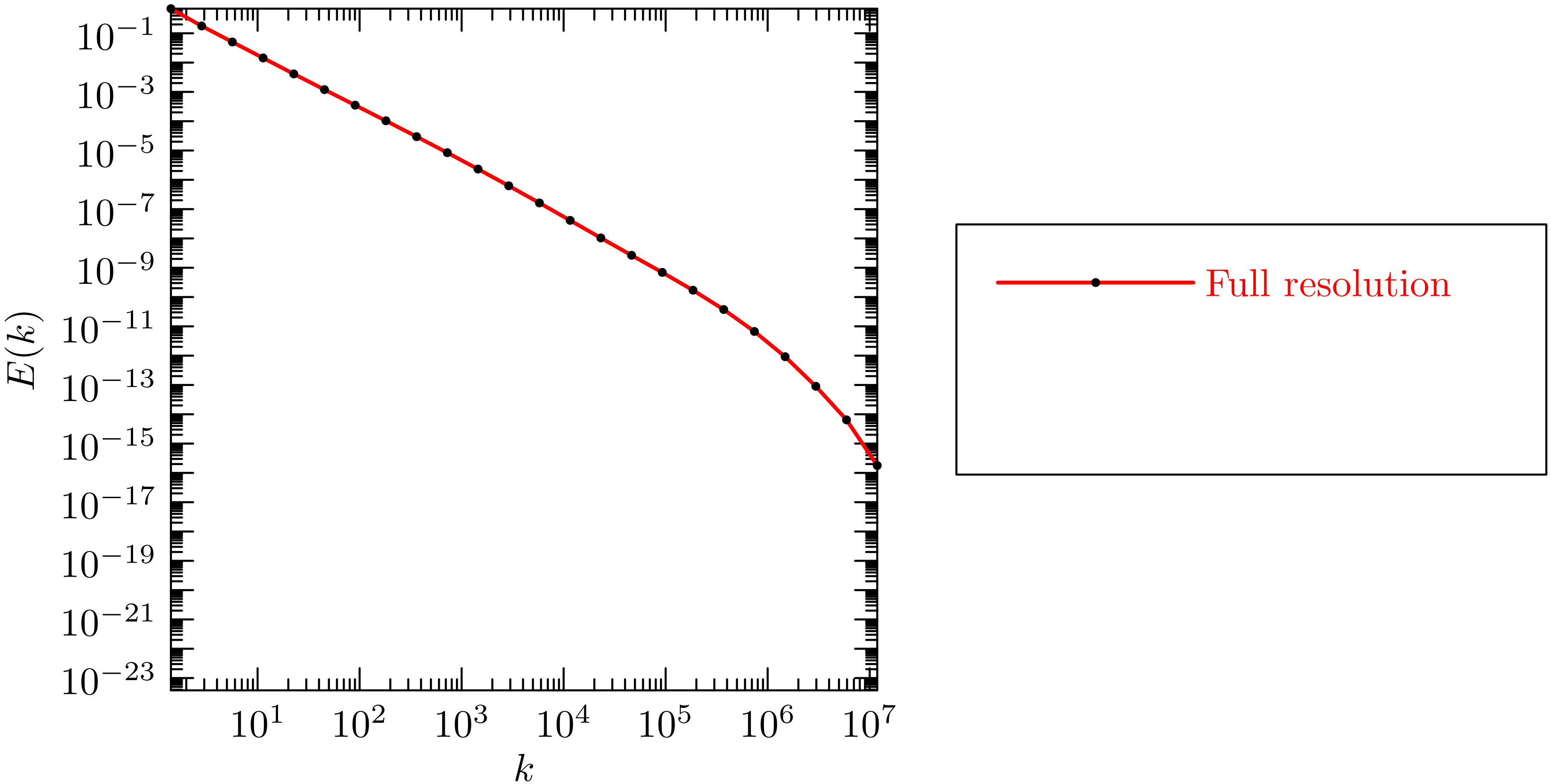

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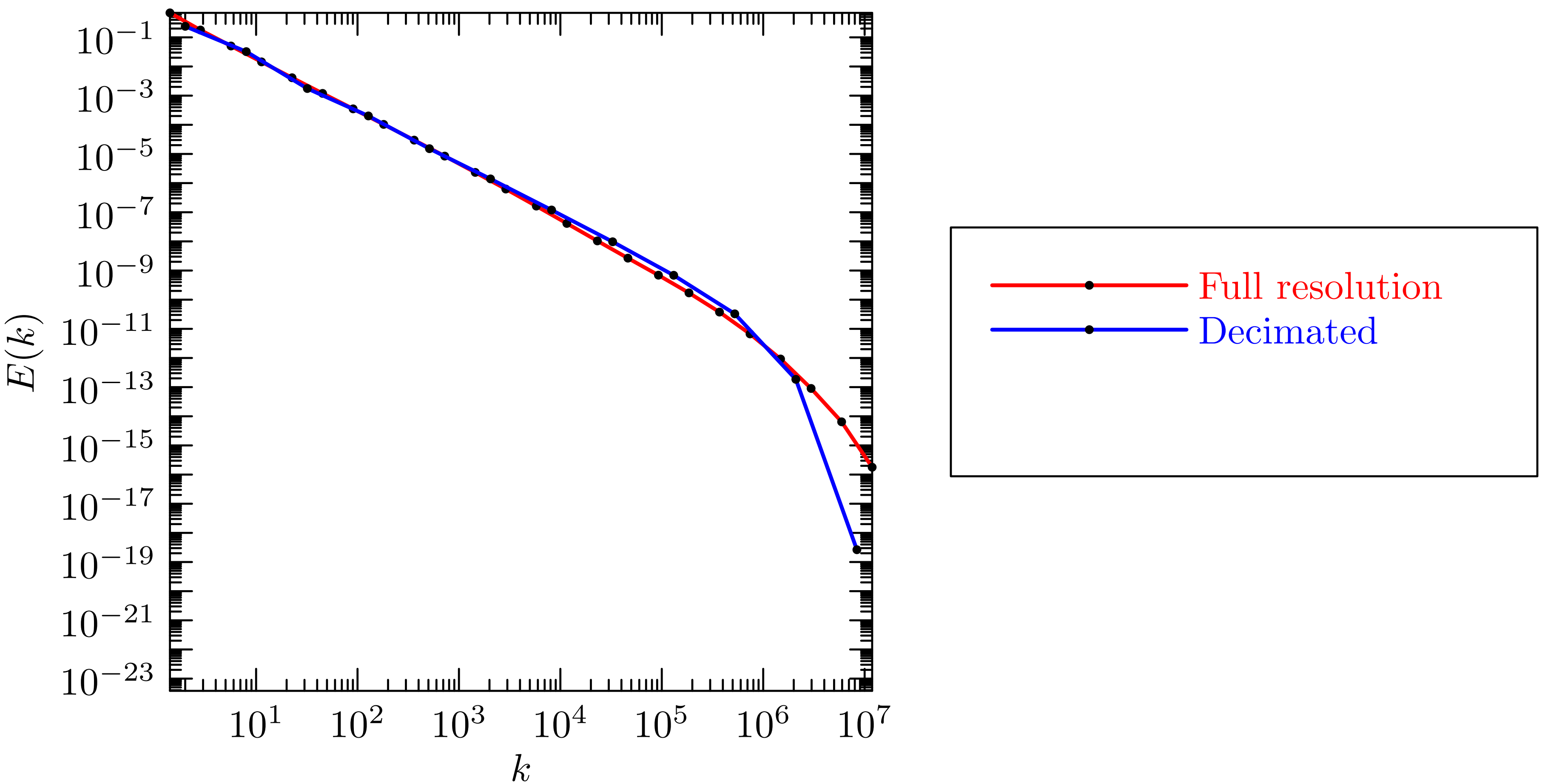

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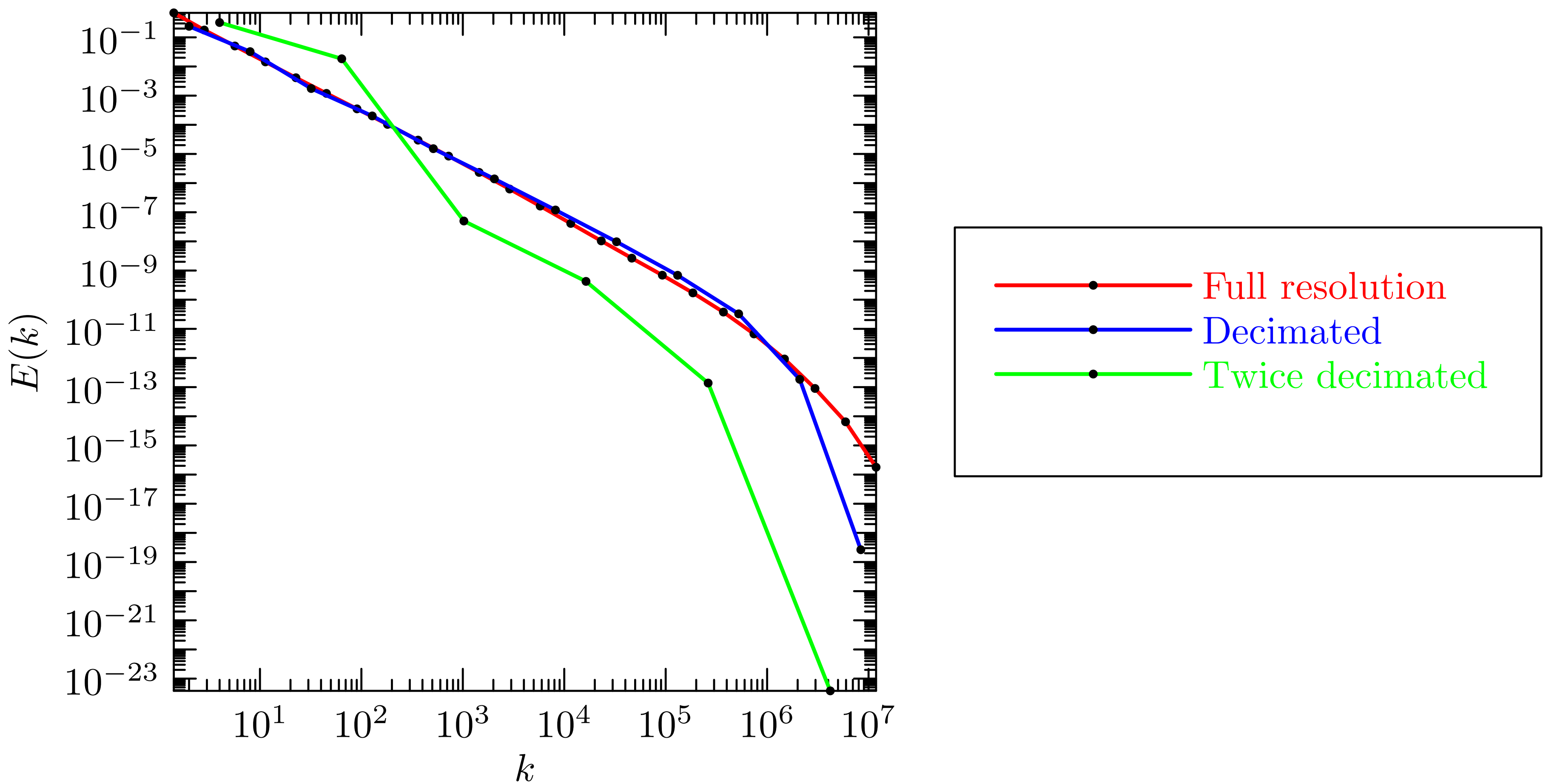

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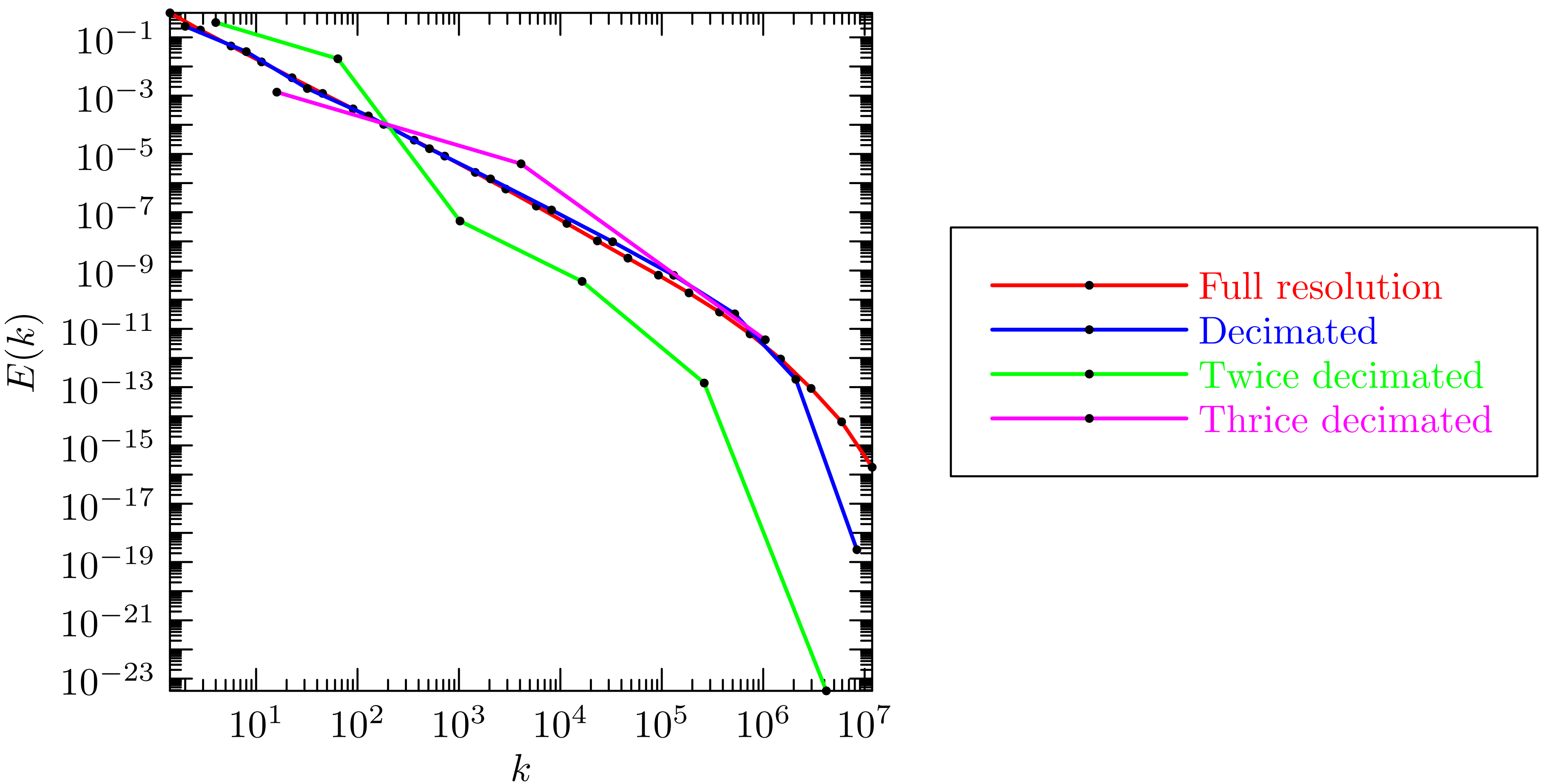

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$$\downarrow$$

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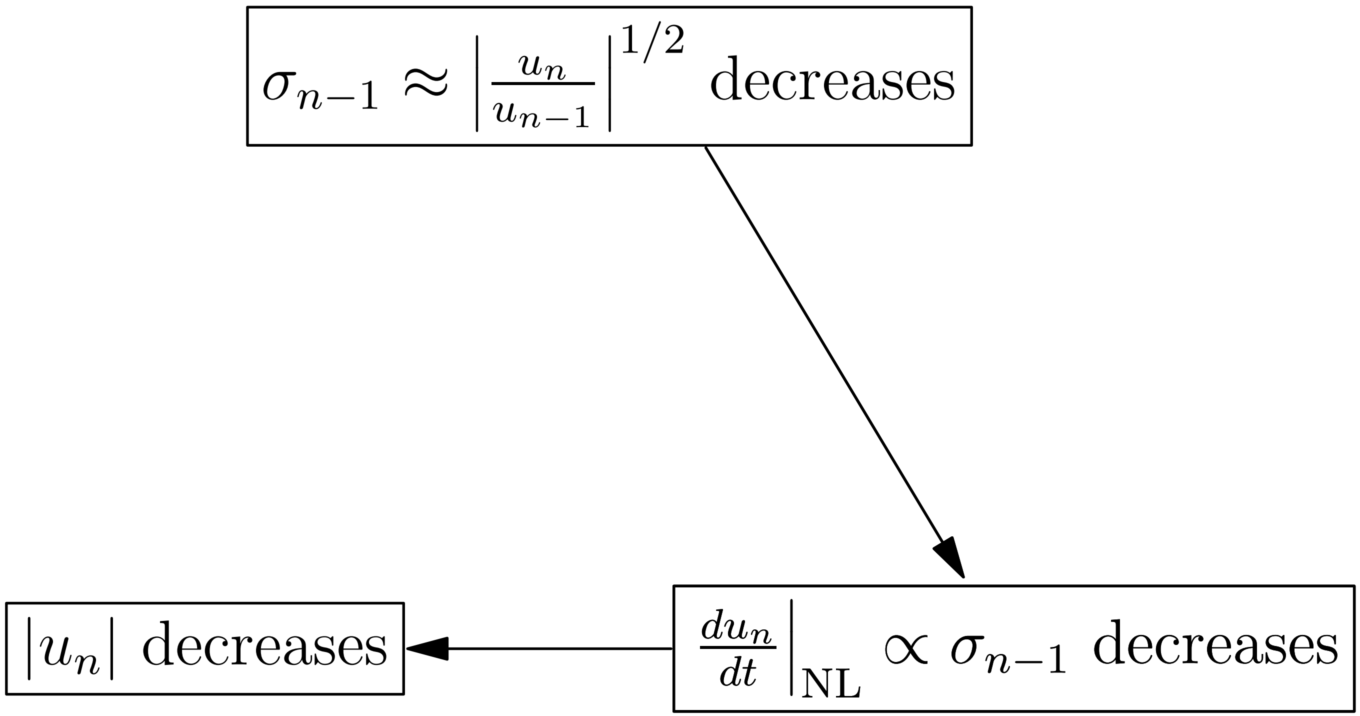

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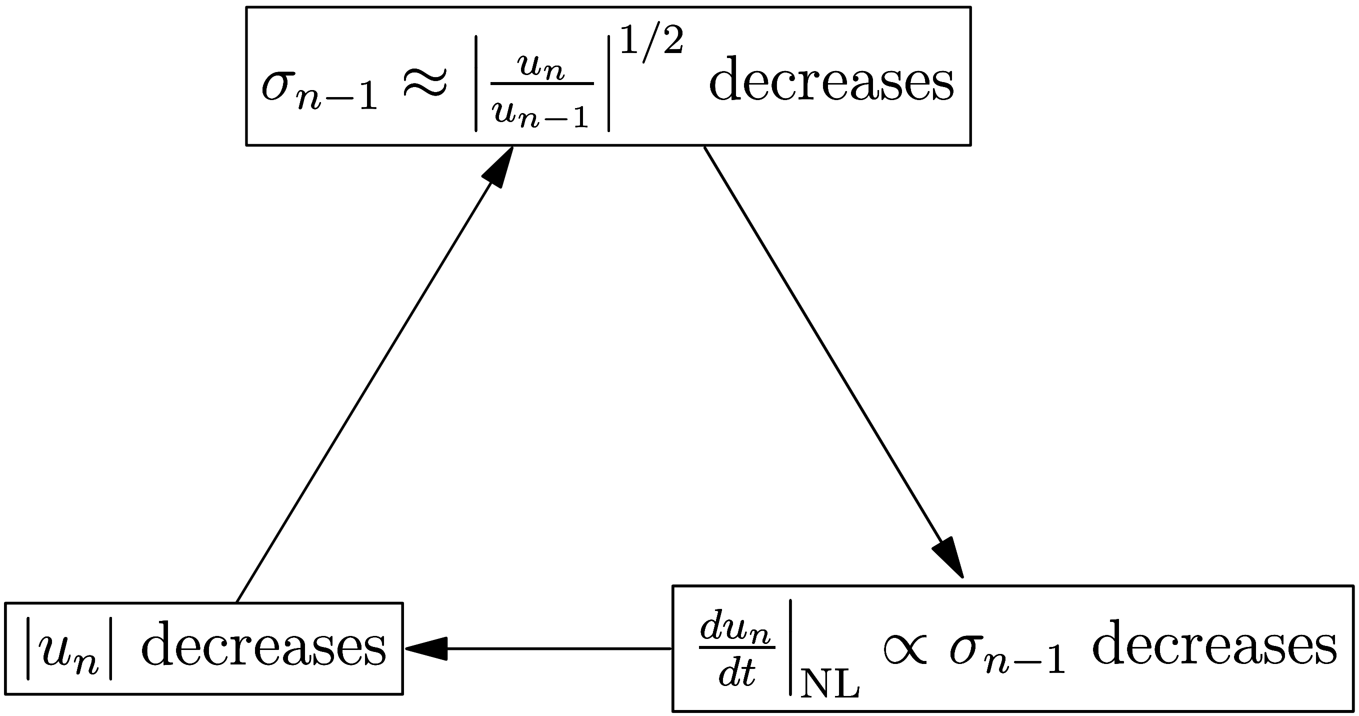

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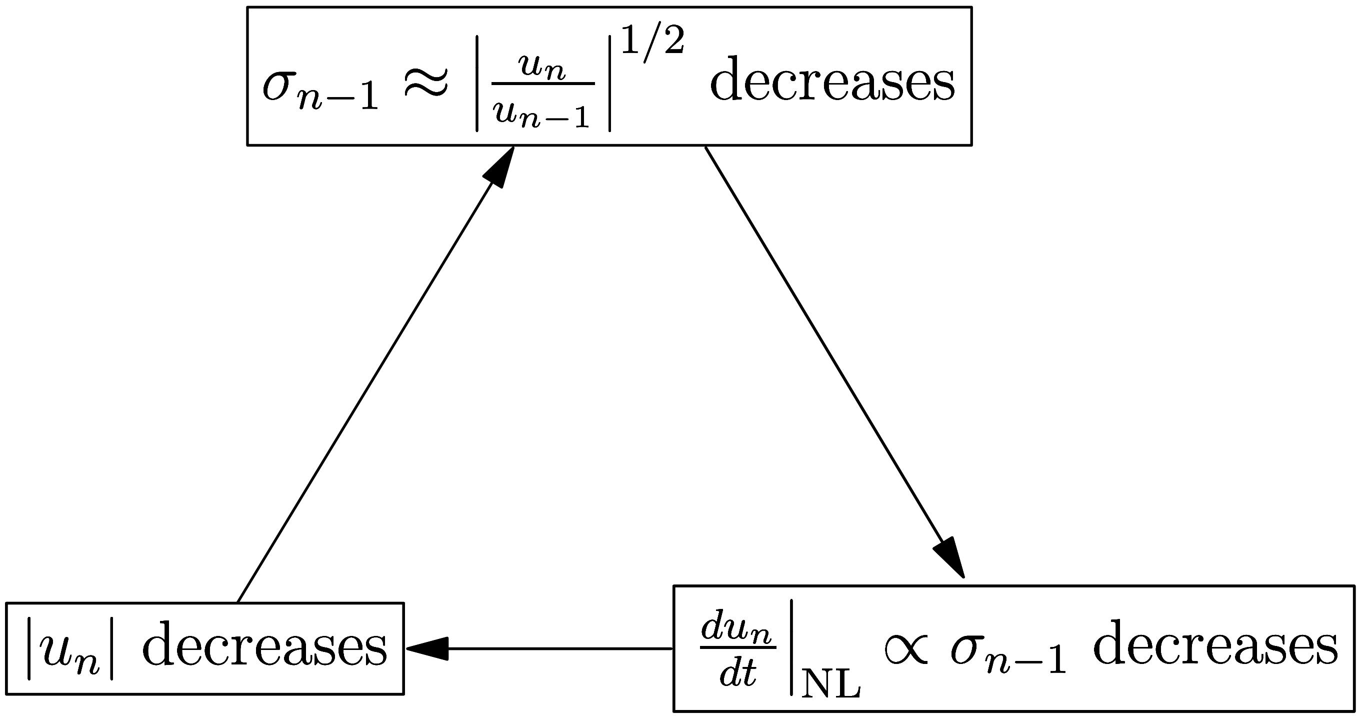

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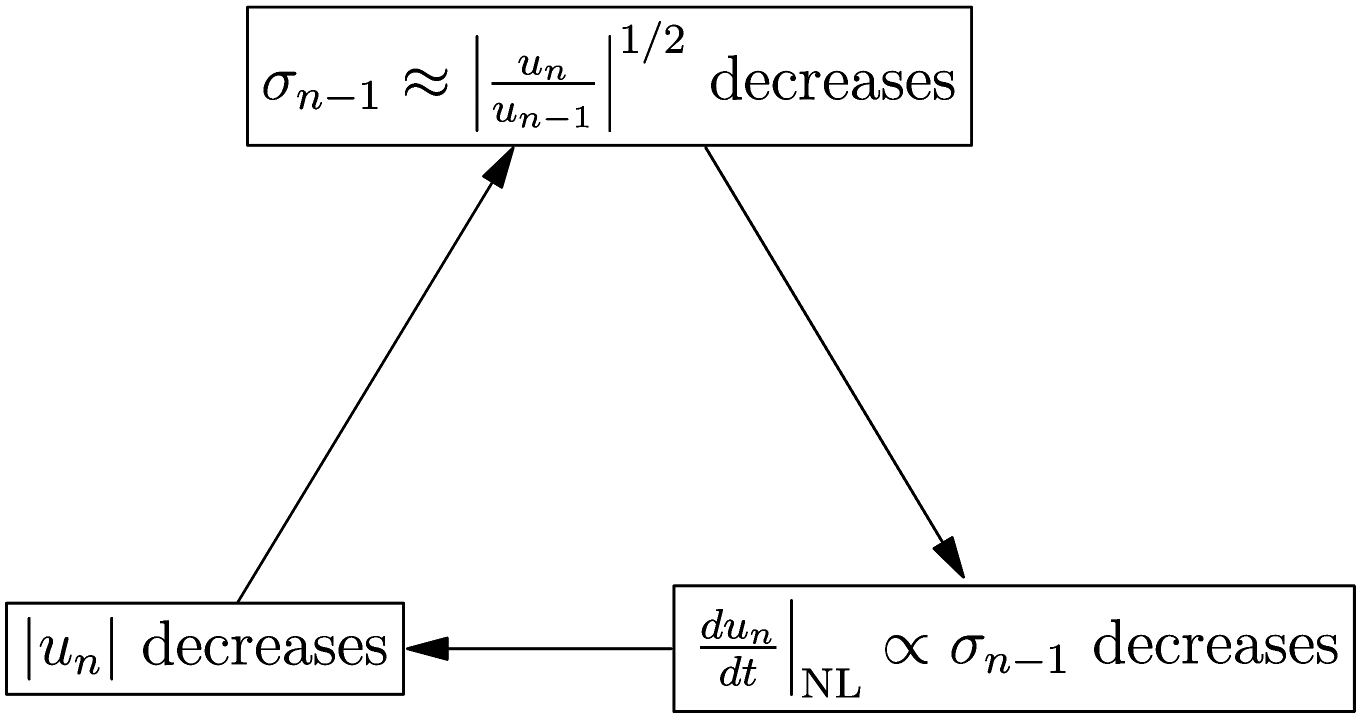

- Energy transfer to mode n is suppressed by positive feedback mechanism!
- We therefore abandon *a posteriori* interpolatation of the unresolved modes and revert to using $\sigma_n = 1$.

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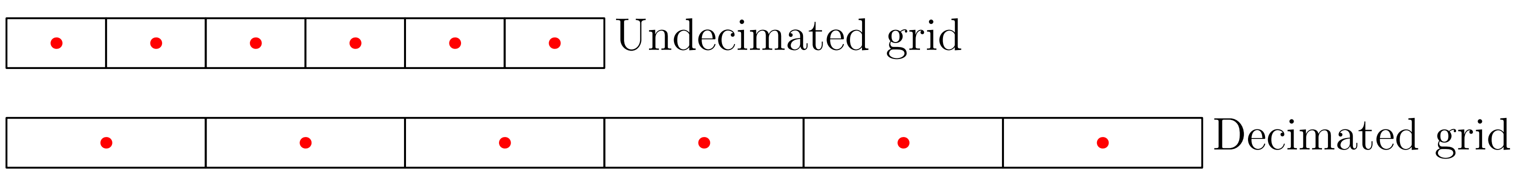

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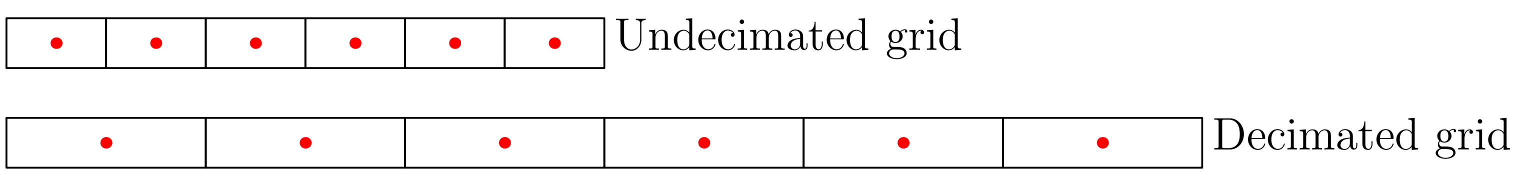
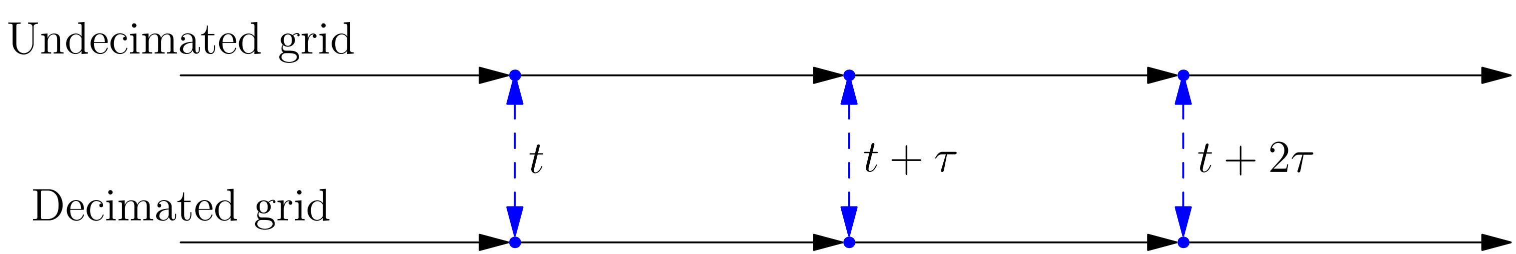


- The grids are advanced using separate integrators and synchronized via projection and prolongation.



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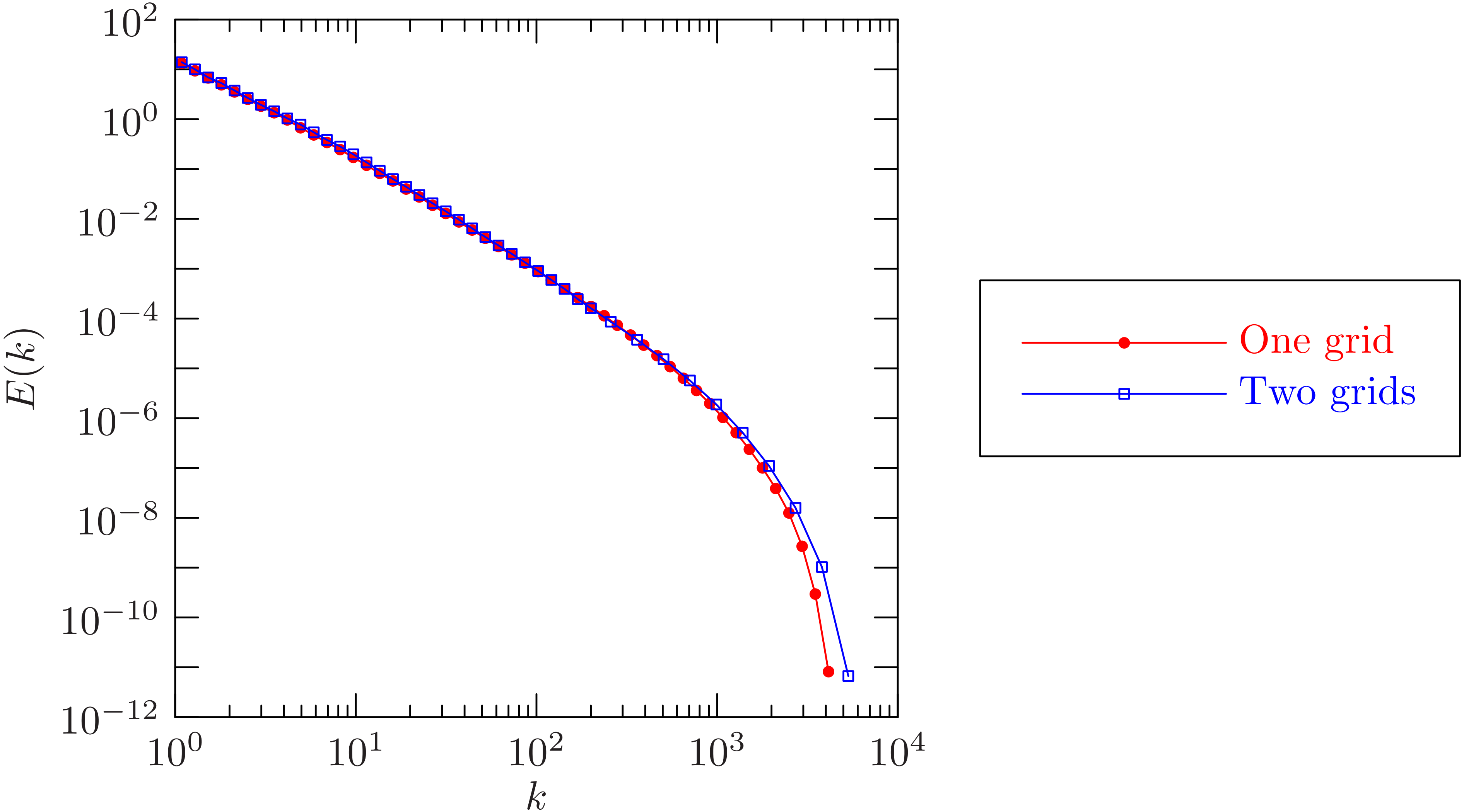

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- The grids must be chosen so that there exist projection and prolongation operators between the grids that locally conserve energy and other quadratic invariants.

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- Piecewise-constant spectral reduction (with $\sigma_n = 1$) has already been applied to 2D NS simulations, but it requires a uniform grid.
- The ultimate goal is to implement the multispectral method for Navier–Stokes turbulence.

References

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