Realizable Markovian Closures

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Fundamental nonlinear equation:

\[
\left( \frac{\partial}{\partial t} + \nu_k \right) \psi_k(t) = \frac{1}{2} \int_{\Delta_k} d\mathbf{p} d\mathbf{q} M_{kpq} \psi_p^* \psi_q^* ,
\]

where \( \int_{\Delta_k} d\mathbf{p} d\mathbf{q} \equiv \int d\mathbf{p} d\mathbf{q} \delta(k + p + q) \).

Symmetrize the mode-coupling coefficients \( M_{kpq} \):

\[
M_{kpq} = M_{kqp}.
\]

For certain real time-independent factors \( \sigma_k \):

\[
\sigma_k M_{kpq} + \sigma_p M_{pqk} + \sigma_q M_{qkp} = 0.
\]
Moments

Define the **two-time correlation**

\[ C_k(t, t') \equiv \langle \psi_k(t) \psi_k^*(t') \rangle \]

and **one-time correlation**

\[ C_k(t) \equiv C_k(t, t) = \left\langle |\psi_k(t)|^2 \right\rangle. \]

The **nonlinear Green’s function** \( G_k(t, t') \) is the infinitesimal response to an added source function \( \bar{\eta}_k \):

\[ G_k(t, t') \equiv \left\langle \frac{\delta \psi_k(t)}{\delta \bar{\eta}_k(t')} \right\rangle \bigg|_{\bar{\eta}_k=0}. \]
Example: $\psi = 2D$ stream function

- Let
  \[ M_{kpq} = \frac{\hat{z} \cdot p \times q}{k^2} (q^2 - p^2) \]

- Conservation of energy
  \[ E = \frac{1}{2} \sum_k k^2 C_k(t) \]
  and enstrophy
  \[ Z = \frac{1}{2} \sum_k k^4 C_k(t) \]

follow, using $\sigma_k = k^2$ and $\sigma_k = k^4$ in
  \[ \sigma_k M_{kpq} + \sigma_p M_{pqk} + \sigma_q M_{qkp} = 0. \]
Closure Tutorial

- Fundamental equation
  \[ \frac{\partial \psi}{\partial t} + \nu \psi = M \psi \psi. \]

- Second moment:
  \[ \frac{\partial \langle \psi \psi \rangle}{\partial t} = 2 \left\langle \frac{\partial \psi}{\partial t} \psi \right\rangle = -2\nu \langle \psi \psi \rangle + 2M \langle \psi \psi \psi \rangle. \]

- Gaussian approximation \( \langle \psi \psi \psi \rangle = 0 \Rightarrow \text{linear theory!} \)

- Instead, formulate the equation for \( \langle \psi \psi \rangle \):
  \[ \frac{\partial}{\partial t} \langle \psi \psi \rangle + 3\nu \langle \psi \psi \rangle = 3M \langle \psi \psi \psi \psi \rangle. \]
Quasinormal Closure

- Adopt Gaussian initial conditions at $t = 0$. Letting $\bar{\psi} = \psi(\bar{t})$,

$$\langle \psi \psi \psi \rangle = 3M \int_0^t d\bar{t} \, e^{-3\nu(t-\bar{t})} \langle \bar{\psi} \bar{\psi} \bar{\psi} \rangle,$$

- Make the quasinormal approximation:

$$\langle \bar{\psi} \bar{\psi} \bar{\psi} \bar{\psi} \rangle = 3 \langle \bar{\psi} \bar{\psi} \rangle \langle \bar{\psi} \bar{\psi} \rangle$$

For Gaussian statistics, this holds exactly.

- We arrive at the quasinormal closure:

$$\frac{\partial}{\partial t} \langle \psi \psi \rangle + 2\nu \langle \psi \psi \rangle = 18MM \int_0^t d\bar{t} \, e^{-3\nu(t-\bar{t})} \langle \bar{\psi} \bar{\psi} \rangle \langle \bar{\psi} \bar{\psi} \rangle.$$

- [Ogura 1963], [Orszag 1977]: the quasinormal closure can incorrectly predict negative energies!
Renormalization

Better to renormalize: replace the linear Green’s function

\[ G^{(0)}(t, \tilde{t}) \equiv e^{-3\nu(t-\tilde{t})} H(t - \tilde{t}), \]

where \( H \) is the Heaviside step function, by the statistical mean \( G \) of the perturbed nonlinear Green’s function \( \tilde{G} \):

\[ \frac{\partial}{\partial t} \tilde{G} + \nu \tilde{G} - 2M \psi \tilde{G} = \delta(t - \tilde{t}). \]

The equations for \( C \overset{\hat{}}{=} \langle \psi \psi \rangle \) and \( G \) take on the form

\[ \frac{\partial}{\partial t} C + 2\nu C = 18MM \int_0^t d\tilde{t} \; G \; C \; C, \]

\[ \frac{\partial}{\partial t} G + \nu G = 9MM \int_0^t d\tilde{t} \; G \; C \; G + \delta(t - \tilde{t}). \]
General form of a closure:

$$\left( \frac{\partial}{\partial t} + \nu_k \right) C_k(t, t') + \int_0^t d\tilde{t} \Sigma_k(t, \tilde{t}) C_k(\tilde{t}, t') = \int_0^{t'} d\tilde{t} \mathcal{F}_k(t, \tilde{t}) G^*_k(t', \tilde{t}),$$

$$\left( \frac{\partial}{\partial t} + \nu_k \right) G_k(t, t') + \int_{t'}^t d\tilde{t} \Sigma_k(t, \tilde{t}) G_k(\tilde{t}, t') = \delta(t - t').$$

Direct-interaction approximation (DIA):

$$\Sigma_k(t, \tilde{t}) = -\int_{\Delta_k} dp \ dq M_{kpq} M^*_{pqk} G^*_p(t, \tilde{t}) C^*_q(t, \tilde{t}),$$

$$\mathcal{F}_k(t, \tilde{t}) = \frac{1}{2} \int_{\Delta_k} dp \ dq M_{kpq} M^*_{kpq} C^*_p(t, \tilde{t}) C^*_q(t, \tilde{t}).$$
Advantages of the DIA

- Reduces correctly to perturbation theory.
- Produces two-time spectral information.
- The DIA can be formally written as
  \[ C_k = G_k \mathcal{F}_k G_k^\dagger, \]
- Whenever \( \mathcal{F}_k \) is a positive definite matrix \( \langle f_k(t)f_k^*(t') \rangle \), the DIA is the exact statistical solution for the generalized Langevin equation
  \[ \left( \frac{\partial}{\partial t} + \nu_k \right) \psi_k(t) + \int_0^t d\tilde{t} \Sigma_k(t, \tilde{t}) \psi_k(\tilde{t}) = f_k(t), \]
Disadvantages of the DIA

- Kramer, Majda, and Vanden-Eijnden [2003] have apparently found a case of passive scalar advection with a fluctuating random sweep where realizability is violated despite the fact that [Kraichnan 1958b] claims the DIA is the exact solution to a random coupling model.
- Violates random Galilean invariance.
- Predicts an energy range $E(k) \sim k^{-3/2}$ instead of $k^{-5/3}$.
- Predicts a 2D enstrophy range $E(k) \sim k^{-5/2}$ instead of $k^{-3}$.
- Contains time-history integrals: nontrivial to compute.
- Only handles second-order statistics; mistreats higher-order coherent structures.
Eddy-Damped QuasiNormal Markovian closure

- The EDQNM approximates the DIA time-history convolutions in favour of a triad interaction time.

**Advantages:**
- Much faster than DIA.
- In the absence of wave phenomena, it is realizable: it predicts the exact statistics of an underlying Langevin equation.

**Disadvantages:**
- Assumes a Fluctuation–Dissipation relation.
- Only predicts one-time spectral information.
- Does not take account of time-history effects accurately.
- Proof of realizability breaks down in the presence of Rossby or drift waves.
- No general multiple-field formulation.
Nonrealizability of the EDQNM
The DIA-based EDQNM

- The DIA equation for the one-time correlation function still contains unknown two-time correlation functions in $\mathcal{F}_k$.
- In thermal equilibrium, the Fluctuation–Dissipation (FD) theorem holds:
  \[ C_k(t, t') = G_k(t, t')C_k(\infty) \quad (t > t'). \]
- In thermal equilibrium, statistical quantities are stationary, so $C_k(t, t') = C_k(t - t')$. Hence $C_k(t) = C_k(0) = C_k(t')$.
- So we must replace the FD theorem by either
  \[ C_k(t, t') = G_k(t, t')C_k(t) \quad (t > t') \]
  or
  \[ C_k(t, t') = G_k(t, t')C_k(t') \quad (t > t'). \]
- EDQNM adopts the 1st form: unlike the 2nd form this leads to a realizable closure [Orszag 1977] in the absence of waves.
In terms of the triad interaction time

\[ \theta_{kpq}(t) \equiv \int_0^t d\tilde{t} \, G_k(t, \tilde{t}) \, G_p(t, \tilde{t}) \, G_q(t, \tilde{t}). \]

the Markovianized DIA can be written in the compact form

\[ \left( \frac{\partial}{\partial t} + 2 \text{Re} \, \nu_k \right) C_k(t) + 2 \text{Re} \, \hat{\eta}_k(t) \, C_k(t) = 2 F_k(t) \]

by defining a nonlinear damping rate,

\[ \hat{\eta}_k(t) \equiv - \int_{\Delta_k} dp \, dq \, M_{kpq} M_{pqk}^* \theta_{kpq}^*(t) \, C_q(t), \]

and a nonlinear noise term,

\[ F_k(t) \equiv \frac{1}{2} \text{Re} \int_{\Delta_k} dp \, dq \, |M_{kpq}|^2 \theta_{kpq}^*(t) \, C_p(t) \, C_q(t). \]
EDQNM

Replace the Green’s function equation by the Markovian form

$$\frac{\partial}{\partial t} G_k(t, t') + \eta_k(t) G_k(t, t') = \delta(t - t'),$$

where $\eta_k(t) \doteq \nu_k + \hat{\eta}_k(t)$. What results is the EDQNM:

$$\frac{\partial}{\partial t} C_k(t) + 2 \text{Re} \eta_k(t) C_k(t) = 2 F_k(t),$$

$$\eta_k(t) \doteq \nu_k - \int_{\Delta_k} dp \, dq \, M_{kpq} M_{pqk}^* \theta_{kpq}^*(t) C_q(t),$$

$$F_k(t) \doteq \frac{1}{2} \text{Re} \int_{\Delta_k} dp \, dq \, |M_{kpq}|^2 \theta_{kpq}^*(t) C_p(t) C_q(t),$$

$$\frac{\partial}{\partial t} \theta_{kpq} + (\eta_k + \eta_p + \eta_q) \theta_{kpq} = 1, \quad \theta_{kpq}(0) = 0.$$  

The computational scaling of this system with time $T$ is $\mathcal{O}T$, a vast improvement over the $\mathcal{O}T^3$ scaling of the DIA.
Test-Field Model (TFM)

- The test-field model [Kraichnan 1971, Kraichnan 1972] also approximates the DIA time-history convolutions.

**Advantages:**
- Invariant to random Galilean transformations.
- Predicts a 2D enstrophy range spectrum $k^{-3}$.
- Much faster than DIA.

**Disadvantages:**
- Heuristic construction.
- Only predicts one-time spectral information.
- Does not take account of time-history effects accurately.
- Assumes a Fluctuation–Dissipation relation.
- Can predict negative energies if wave effects are present!
Nonrealizability of the EDQNM and TFM
Realizable Markovian Closure (RMC)

- Goal: Replace the FD Ansatz with a relation that reduces to the FD theorem in a steady state.

- EDQNM FD Ansatz:
  \[
  \frac{C_k(t, t')}{C_k(t)} = G_k(t, t') \quad (t \geq t'),
  \]

- Langevin statistics:
  \[
  \frac{C_k(t, t')}{C_k(t')} = G_k(t, t') \quad (t \geq t').
  \]

- Thermal equilibrium:
  \[
  \frac{C_k(t, t')}{C_k(\infty)} = G_k(t, t') \quad (t \geq t').
  \]
Modified Fluctuation–Dissipation Ansatz

In a non stationary state, (19) should be restated as a balance between the correlation coefficient and the response function (for $t \geq t'$):

\[
\frac{C_k(t, t')}{{C_k}^{1/2}(t) {C_k}^{1/2}(t')} = \frac{C_{1/2}^{1/2}(t) C_{1/2}^{1/2}(t')}{G_k(t, t')}.
\]

Time scales of amplitude decorrelation and decay of infinitesimal disturbances should be equal, since these processes both occur by interaction with the turbulent background.
Realizability

For unrestricted time arguments $t$ and $t'$:

$$C_k(t, t') = C_k^{1/2}(t) \left[ G_k(t, t') + G_k^*(t', t) \right] C_k^{1/2*}(t').$$

$C_k(t, t')$ positive-semidefinite $\iff$ $G^h_k(t, t') = G_k(t, t') + G_k^*(t', t)$ is positive-semidefinite.

We employ the following theorem [Bowman et al. 1993]:

**Theorem 1:** Let $\text{Re} \eta_k(t)$ be continuous almost everywhere.

The Hermitian function $G^h_k$ defined by

$$G^h_k(t, t') \doteq \begin{cases} e^{-\int_t^t \eta_k(\bar{t})d\bar{t}}, & \text{for } t \geq t'; \\ e^{-\int_t^{t'} \eta_k^*(\bar{t})d\bar{t}}, & \text{for } t < t', \end{cases}$$

is positive-semidefinite $\iff$ $\text{Re} \eta_k(t) \geq 0$ for almost all $t$. 
Subject to the restriction $\text{Re} \eta_k \geq 0$, it follows that

$$C_k(t, t) = \int d\tilde{t} d\tilde{t} G_k(t, \tilde{t}) \mathcal{F}_k(\tilde{t}, \tilde{t}) G^*_k(t, \tilde{t}) \geq 0,$$

is real and non-negative, provided that the initial condition is non-negative.
Realizable Markovian Closure (RMC)

Applying the modified FD Ansatz yields the RMC:

\[ \frac{\partial}{\partial t} C_k(t) + 2 \text{Re} \eta_k(t) C_k(t) = 2F_k(t) \]

\[ \eta_k(t) \doteq \nu_k - \sum_{k+p+q=0} M_{kpq} M^*_{pqk} \Theta^*_{pqk}(t) C_q^{1/2}(t) C_k^{-1/2}(t) \]

\[ 2F_k(t) \doteq \text{Re} \sum_{k+p+q=0} |M_{kpq}|^2 \Theta_{kpq}(t) C_p^{1/2}(t) C_q^{1/2}(t) \]

\[ \frac{\partial}{\partial t} \Theta_{kpq} + [\eta_k + \mathcal{P}(\eta_p) + \mathcal{P}(\eta_q)] \Theta_{kpq} = C_p^{1/2} C_q^{1/2}, \]

where \( \mathcal{P}(\eta) \doteq \text{Re} \eta \ H(\text{Re} \eta) + i \ \text{Im} \ \eta \) and \( H \) is the Heaviside unit step function.

Although the steady-state EDQNM and RMC equations are identical, the RMC provides a realizable evolution to this state.
Similarly, we have constructed a Realizable Test-Field Model [Bowman & Krommes 1997]. Even for wave-free turbulence, the RMC and RTFM appear to be more representative of the true dynamics than the EDQNM and TFM.

The RMC and RTFM possess underlying Langevin equations:

$$\frac{\partial}{\partial t}\psi + \eta \psi = f,$$

which, unlike the EDQNM, does not assume $\delta$-correlated statistics.

It is also possible to design multiple-rate Markovian closures that allow for different decorrelation and infinitesimal perturbation decay rates; this may afford a more accurate treatment of non-white noise effects.
Comparison of RMC and RTFM with DNS

![Graph showing comparison of RMC, RTFM, and DNS]

- **RMC**
- **RTFM**
- **DNS**
Alternatives

- Mapping Closures
- Kaneda’s Lagrangian Renormalized Approximation (LRA) [Kaneda 1981]
- McComb’s Local Energy Theory (LET) [McComb 1990]
- Direct Numerical Simulation
- Dynamic Subgrid Models
- Renormalization Group Theory
- Reduced Models:
  - Decimation
  - Empirical Orthogonal Eigenfunctions
  - Spectral Reduction: Bowman, Shadwick, Morrison [1999]
  - Stochastic Models
2D Turbulence

- Energy $E = \frac{1}{2} \sum_{k} \frac{|\omega_k|^2}{k^2}$ and enstrophy $Z = \frac{1}{2} \sum_{k} |\omega_k|^2$ are conserved.

\[ E_2 = E_1 + E_3, \quad Z_2 = Z_1 + Z_3. \]

- [Fjørtoft 1953]: energy cascades to large scales and enstrophy cascades to small scales.

- [Kraichnan 1967], [Leith 1968], and [Batchelor 1969] (KLB): $k^{-5/3}$ inverse energy cascade at large scales, $k^{-3}$ direct enstrophy cascade at small scales.
Let $s^2 = \frac{\sum_k f_k \omega_k^*}{\sum_k f_k \frac{\omega_k^*}{k^2}}$ be the ratio of mean enstrophy to energy injection.

Typically, $s$ will lie within the band of forced wavenumbers.

Multiply the energy equation

$$
\frac{1}{2k^2} \frac{\partial |\omega_k|^2}{\partial t} + D_k \frac{|\omega_k|^2}{k^2} = S_k \frac{\omega_k^*}{k^2} + f_k \frac{\omega_k^*}{k^2}
$$

by $s^2$ and subtract the enstrophy equation

$$
\frac{1}{2} \frac{\partial |\omega_k|^2}{\partial t} + D_k |\omega_k|^2 = S_k \omega_k^* + f_k \omega_k^*
$$

$\Rightarrow$ steady-state balance equation [Tran & Bowman 2003]

$$
\sum_{k=1}^{s} (s^2 - k^2) D_k E(k) = \sum_{k=s}^{\infty} (k^2 - s^2) D_k E(k).
$$
Balance Equation

- Small and large scale dynamics are intricately coupled:
  \[
  \sum_{k=1}^{s} (s^2 - k^2) D_k E(k) = \sum_{k=s}^{\infty} (k^2 - s^2) D_k E(k).
  \]

- Can be used to explain the discrepancy between the KLB prediction \( E(k) \sim k^{-3} \) and the steep \( \sim k^{-5} \) enstrophy-range spectrum typically seen in numerical simulations.

- Unbounded domain: everlasting inverse energy cascade.

- Bounded domain: upscale energy cascade is halted at the lowest wavenumber.

- The effect of this lower spectral boundary may be understood by replacing it with an external forcing.
Large-scale direct cascade (zero dissipation for $k < 40$)?
Energetic reflections at the lower spectral boundary eventually lead to a large-scale direct cascade.

This would agree with the large-scale $k^{-3}$ spectra seen numerically by [Borue 1994] and observed in the atmosphere [Lilly & Peterson 1983].

[Tran & Bowman 2003]: In a bounded domain, the two inertial range exponents must sum to $-8$ (high Reynolds number).

Large-scale $k^{-3}$ spectrum $\Rightarrow$ a small-scale $k^{-5}$ spectrum.

Consistent with rigorous [Tran & Shepherd 2002] constraint: the spectrum must be at least as steep as $k^{-5}$. 
Direct $k^{-3}$ enstrophy cascade

Zero dissipation for $3 < k < 300$. 
Logarithmic spectral slope

Zero dissipation for $3 < k < 300$. 

Logarithmic slope of $E(k)$
Conclusions

- **Realizability** ensures physically reasonable behaviour.
- The EDQNM closure can predict negative energies in the presence of non-hermitian effects such as wave phenomena.
- The unrealizability of the EDQNM closure arises from an improper Fluctuation–Dissipation Ansatz.
- Correcting this difficulty has led to the realizable Markovian closure.
- A realizable test-field model, invariant to random Galilean transformations, has been implemented for two-dimensional Navier–Stokes turbulence.
## References


