Spectral Reduction: A Statistical Description of Turbulence

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Outline

1 Spectral Reduction

A Properties

B Moments

C Liouville Theorem

D Statistical Equipartition

2 2D Enstrophy Cascade

A Comparison with Full Simulation

B Structure Functions

C Coherent Structures

3 Conclusions

2D Turbulence

- 2D Navier–Stokes vorticity equation:

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} = \int d\boldsymbol{p} \int d\boldsymbol{q}\, \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^*,$$

  where $\nu_{\boldsymbol{k}} \doteq \nu k^2$ and

$$\epsilon_{\boldsymbol{kpq}} \doteq (\hat{\boldsymbol{z}} \cdot \boldsymbol{p} \times \boldsymbol{q})\, \delta(\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q})$$

  is antisymmetric under permutation of any two indices.

- Energy E_0 and enstrophy Z_0 on the fine grid:

$$E_0 \doteq \frac{1}{2} \int d\boldsymbol{k}\, \frac{|\omega_{\boldsymbol{k}}|^2}{k^2}, \qquad Z_0 \doteq \frac{1}{2} \int d\boldsymbol{k}\, |\omega_{\boldsymbol{k}}|^2 .$$

- First consider $\nu_{\boldsymbol{k}} = 0$. Conservation of E_0 and Z_0 follow from:

$$\frac{1}{k^2} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \quad \text{antisymmetric in} \quad \boldsymbol{k} \leftrightarrow \boldsymbol{q},$$

$$\frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \quad \text{antisymmetric in} \quad \boldsymbol{k} \leftrightarrow \boldsymbol{p}.$$

Spectral Reduction

- Introduce a coarse-grained grid indexed by K.
- Define new variables

$$\Omega_{\boldsymbol{K}} = \langle \omega_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \doteq \frac{1}{\Delta_{\boldsymbol{K}}} \int_{\Delta_{\boldsymbol{K}}} \omega_{\boldsymbol{k}} \, d\boldsymbol{k},$$

where $\Delta_{\boldsymbol{K}}$ is the area of bin $\boldsymbol{K}$.

- Evolution of $\Omega_{\boldsymbol{K}}$:

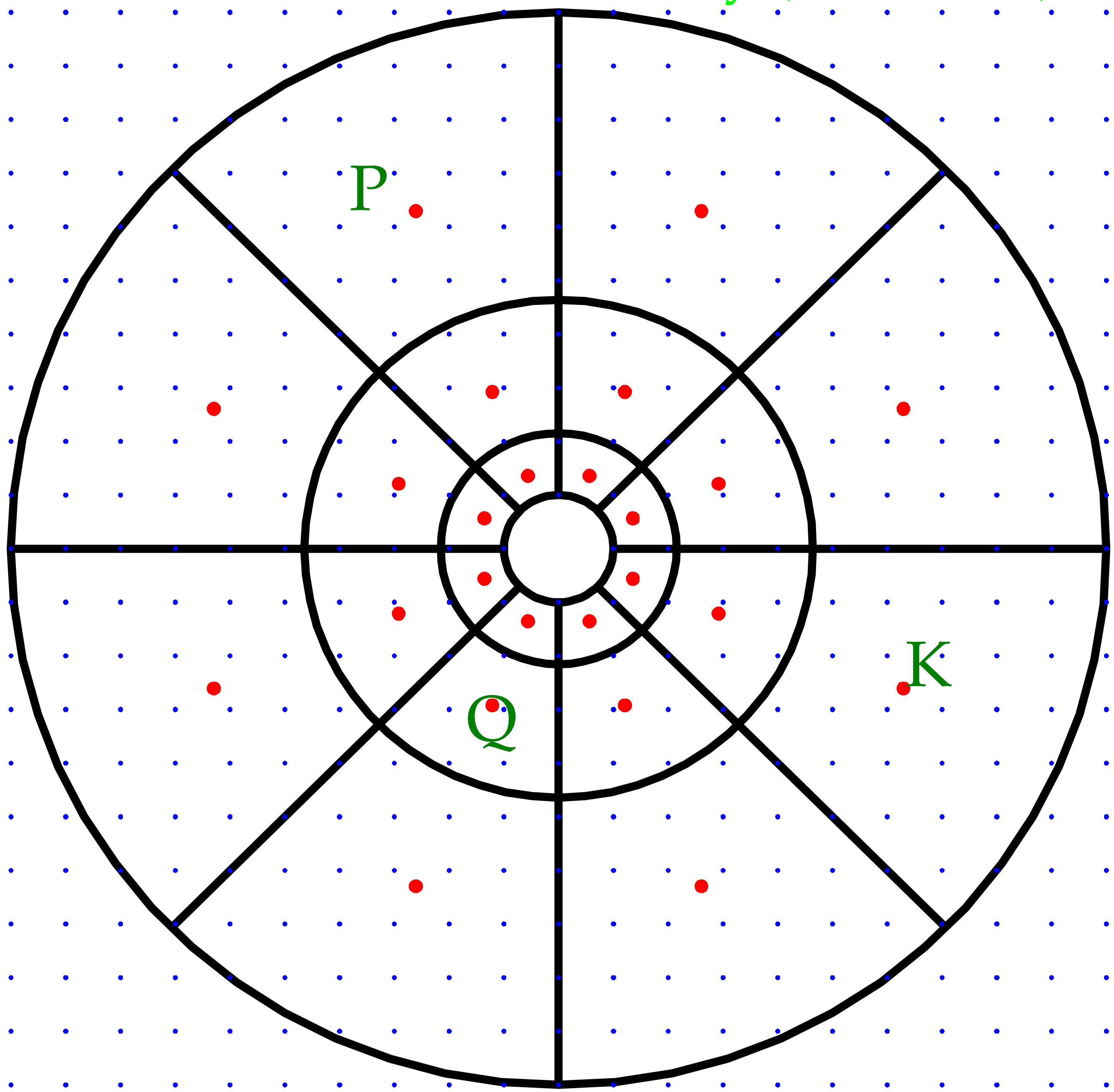
$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^* \right\rangle_{\boldsymbol{KPQ}},$$

where $\langle f \rangle_{\boldsymbol{KPQ}} = \dfrac{1}{\Delta_{\boldsymbol{K}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}}} \displaystyle\int_{\Delta_{\boldsymbol{K}}} d\boldsymbol{k} \int_{\Delta_{\boldsymbol{P}}} d\boldsymbol{p} \int_{\Delta_{\boldsymbol{Q}}} d\boldsymbol{q} \, f.$

- Approximate $\omega_{\boldsymbol{p}}$ and $\omega_{\boldsymbol{q}}$ by bin-averaged values $\Omega_{\boldsymbol{P}}$ and $\Omega_{\boldsymbol{Q}}$:

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \Omega_{\boldsymbol{P}}^* \Omega_{\boldsymbol{Q}}^*.$$

Wavenumber Bin Geometry (3 x 8 bins)



- On the coarse grid, define the energy E and enstrophy Z

$$E \doteq \frac{1}{2}\sum_{\boldsymbol{K}} \frac{|\Omega_{\boldsymbol{K}}|^2}{K^2}\Delta_{\boldsymbol{K}}, \qquad Z \doteq \frac{1}{2}\sum_{\boldsymbol{K}} |\Omega_{\boldsymbol{K}}|^2 \Delta_{\boldsymbol{K}}.$$

- Enstrophy is still conserved since

$$\left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \qquad \text{antisymmetric in} \qquad \boldsymbol{K} \leftrightarrow \boldsymbol{P}.$$

- But energy conservation has been lost!

$$\frac{1}{K^2}\left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \qquad \text{NOT antisymmetric in} \qquad \boldsymbol{K} \leftrightarrow \boldsymbol{Q}.$$

- Reinstate both desired symmetries with the modified coefficient

$$\frac{\langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}}}{Q^2}.$$

- Energy and enstrophy are now simultaneously conserved.

Properties

- We call the forced-dissipative version of this approximation *Spectral Reduction* (SR):

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \frac{\left\langle \epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{KPQ}}}{Q^2} \Omega^*_{\boldsymbol{P}} \Omega^*_{\boldsymbol{Q}}.$$

- SR conserves both energy and enstrophy and reduces to the exact dynamics in the limit of small bin size.
- It has the same general structure and symmetries as the original equation and in this sense may be considered a *renormalization*.
- SR obeys a Liouville Theorem; in the inviscid limit, it yields statistical-mechanical (equipartition) solutions.

Moments

- Q. How accurate is Spectral Reduction?
- A. For large bins, the *instantaneous* dynamics of SR is inaccurate.
- However: the equations for the *time-averaged* (or ensemble-averaged) moments predicted by SR closely approximate those of the exact bin-averaged statistics.
  Eg., time average the exact bin-averaged enstrophy equation:

$$\overline{\frac{\partial}{\partial t}\left\langle |\omega_{\boldsymbol{k}}|^2 \right\rangle_{\boldsymbol{K}}} + 2\,\mathrm{Re}\left\langle \nu_{\boldsymbol{k}} \overline{|\omega_{\boldsymbol{k}}|^2} \right\rangle_{\boldsymbol{K}} = 2\,\mathrm{Re} \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \overline{\omega_{\boldsymbol{k}}^* \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^*} \right\rangle_{\boldsymbol{KPQ}},$$

  where the bar means time average and $\langle \cdot \rangle_{\boldsymbol{K}}$ means bin average.

- Time-averaged quantities such as $\overline{|\omega_{\boldsymbol{k}}|^2}$ and $\overline{\omega_{\boldsymbol{k}}^* \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^*}$ are generally *smooth* functions of $\boldsymbol{k}$, $\boldsymbol{p}$, $\boldsymbol{q}$ on the four-dimensional surface defined by the triad condition $\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q} = 0$.

- Mean Value Theorem for integrals: for some $\boldsymbol{\xi} \in \boldsymbol{K}$,

$$\overline{|\Omega_{\boldsymbol{K}}|^2} = \overline{|\omega_{\boldsymbol{\xi}}|^2} \approx \overline{|\omega_{\boldsymbol{k}}|^2} \qquad \forall \boldsymbol{k} \in \boldsymbol{K}.$$

- To good accuracy these statistical moments may therefore be evaluated at the characteristic wavenumbers $\boldsymbol{K}$, $\boldsymbol{P}$, $\boldsymbol{Q}$:

$$\overline{\frac{\partial}{\partial t}|\Omega_{\boldsymbol{K}}|^2} + 2\,\mathrm{Re}\,\langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \overline{|\Omega_{\boldsymbol{K}}|^2} = 2\,\mathrm{Re} \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \overline{\Omega^*_{\boldsymbol{K}} \Omega^*_{\boldsymbol{P}} \Omega^*_{\boldsymbol{Q}}}.$$

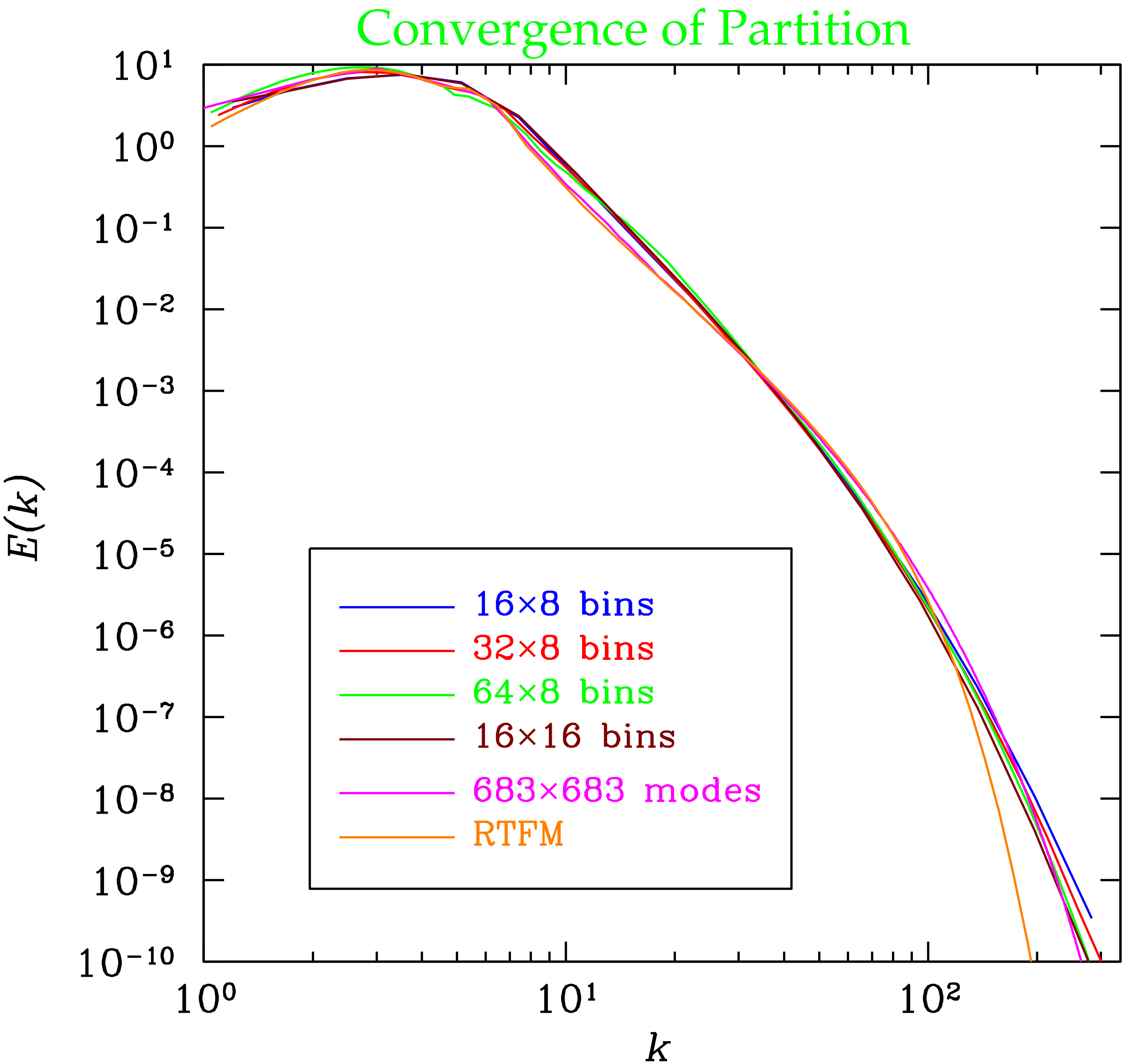
To the extent that the wavenumber magnitude q varies slowly over a bin:

$$\overline{\frac{\partial}{\partial t}|\Omega_{\boldsymbol{K}}|^2} + 2\,\mathrm{Re}\,\langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \overline{|\Omega_{\boldsymbol{K}}|^2} = 2\,\mathrm{Re} \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \frac{\langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}}}{Q^2} \overline{\Omega^*_{\boldsymbol{K}} \Omega^*_{\boldsymbol{P}} \Omega^*_{\boldsymbol{Q}}}.$$

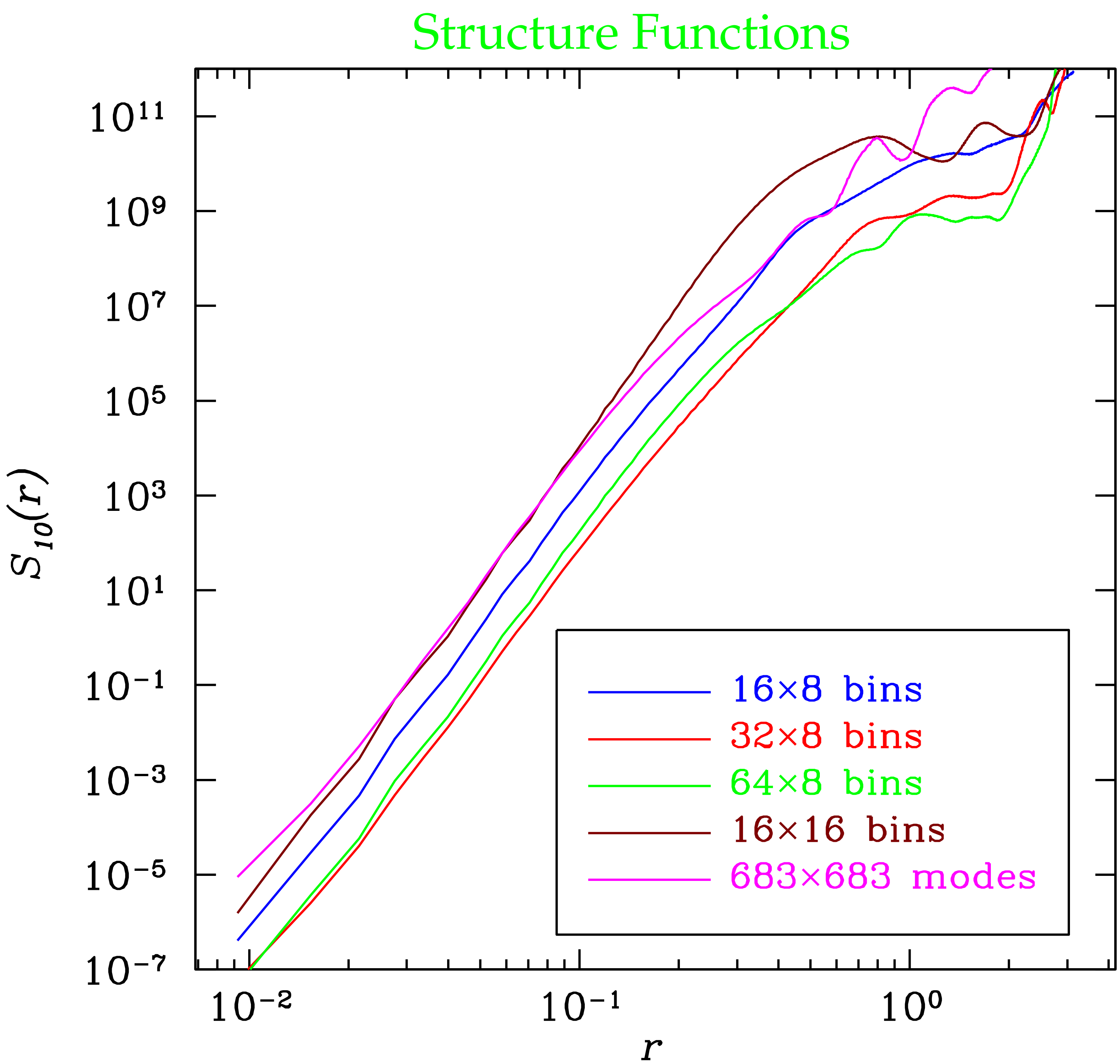
- But this is precisely the time-average of the SR equation!

Convergence

- The previous argument suggests that Spectral Reduction can indeed provide an accurate statistical description of turbulence, even when each bin contains many statistically independent modes.
- As the wavenumber partition is refined, one expects the solutions of the time-averaged SR equations to converge to the exact statistical solution.
- An object-oriented C^{++} program `(Triad)` has been developed to implement and test Spectral Reduction.



Convergence of Partition
10¹
10⁰
10⁻¹
10⁻²
10⁻³
10⁻⁴
10⁻⁵
10⁻⁶
10⁻⁷
10⁻⁸
10⁻⁹
10⁻¹⁰
E(k)
16×8 bins
32×8 bins
64×8 bins
16×16 bins
683×683 modes
RTFM
10⁰
10¹
10²
k



Structure Functions
$S_{10}(r)$
r
10^{11}
10^{9}
10^{7}
10^{5}
10^{3}
10^{1}
10^{-1}
10^{-3}
10^{-5}
10^{-7}
10^{-2}
10^{-1}
10^{0}
16×8 bins
32×8 bins
64×8 bins
16×16 bins
683×683 modes

Noncanonical Hamiltonian Formulation

- Underlying *noncanonical* Hamiltonian formulation for inviscid 2D vorticity equation:

$$\dot{\omega}_{\boldsymbol{k}} = \int d\boldsymbol{q}\, J_{\boldsymbol{k}\boldsymbol{q}} \frac{\delta H}{\delta \omega_{\boldsymbol{q}}},$$

where

$$H \doteq \frac{1}{2} \int d\boldsymbol{k}\, \frac{|\omega_{\boldsymbol{k}}|^2}{k^2},$$

$$J_{\boldsymbol{k}\boldsymbol{q}} \doteq \int d\boldsymbol{p}\, \epsilon_{\boldsymbol{k}\boldsymbol{p}\boldsymbol{q}} \omega^*_{\boldsymbol{p}}.$$

- Leads to inviscid Navier–Stokes equation:

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} = \int d\boldsymbol{p} \int d\boldsymbol{q}\, \frac{\epsilon_{\boldsymbol{k}\boldsymbol{p}\boldsymbol{q}}}{q^2} \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}}.$$

Liouville Theorem

- Navier–Stokes:

$$J_{\boldsymbol{kq}} \doteq \int d\boldsymbol{p}\, \epsilon_{\boldsymbol{kpq}} \omega_{\boldsymbol{p}}^*$$

$$\Rightarrow \qquad \int d\boldsymbol{k}\, \frac{\delta \dot{\omega}_{\boldsymbol{k}}}{\delta \omega_{\boldsymbol{k}}} = \int d\boldsymbol{k} \int d\boldsymbol{q}\, \underbrace{\frac{\delta J_{\boldsymbol{kq}}}{\delta \omega_{\boldsymbol{k}}}} \frac{\delta H}{\delta \omega_{\boldsymbol{q}}} + J_{\boldsymbol{kq}} \frac{\delta^2 H}{\delta \omega_{\boldsymbol{k}} \delta \omega_{\boldsymbol{q}}} = 0.$$

$$\epsilon_{\boldsymbol{k}(-\boldsymbol{k})\boldsymbol{q}} = 0$$

- Spectral Reduction:

$$J_{\boldsymbol{KQ}} \doteq \sum_{\boldsymbol{P}} \Delta_{\boldsymbol{P}} \left\langle \epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{KPQ}} \Omega_{\boldsymbol{P}}^*$$

$$\Rightarrow \qquad \sum_{\boldsymbol{K}} \frac{\partial \dot{\Omega}_{\boldsymbol{K}}}{\partial \Omega_{\boldsymbol{K}}} = \sum_{\boldsymbol{K},\boldsymbol{Q}} \underbrace{\frac{\partial J_{\boldsymbol{KQ}}}{\partial \Omega_{\boldsymbol{K}}}} \frac{\partial H}{\partial \Omega_{\boldsymbol{Q}}} + J_{\boldsymbol{KQ}} \frac{\partial^2 H}{\partial \Omega_{\boldsymbol{K}} \partial \Omega_{\boldsymbol{Q}}} = 0.$$

$$\left\langle \epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{K}(-\boldsymbol{K})\boldsymbol{Q}} = 0$$

Statistical Equipartition

- If the dynamics are *mixing*, the Liouville Theorem and the coarse-grained invariants

$$E \doteq \frac{1}{2}\sum_{\boldsymbol{K}} \frac{|\Omega_{\boldsymbol{K}}|^2}{K^2}\Delta_{\boldsymbol{K}}, \qquad Z \doteq \frac{1}{2}\sum_{\boldsymbol{K}} |\Omega_{\boldsymbol{K}}|^2 \Delta_{\boldsymbol{K}},$$

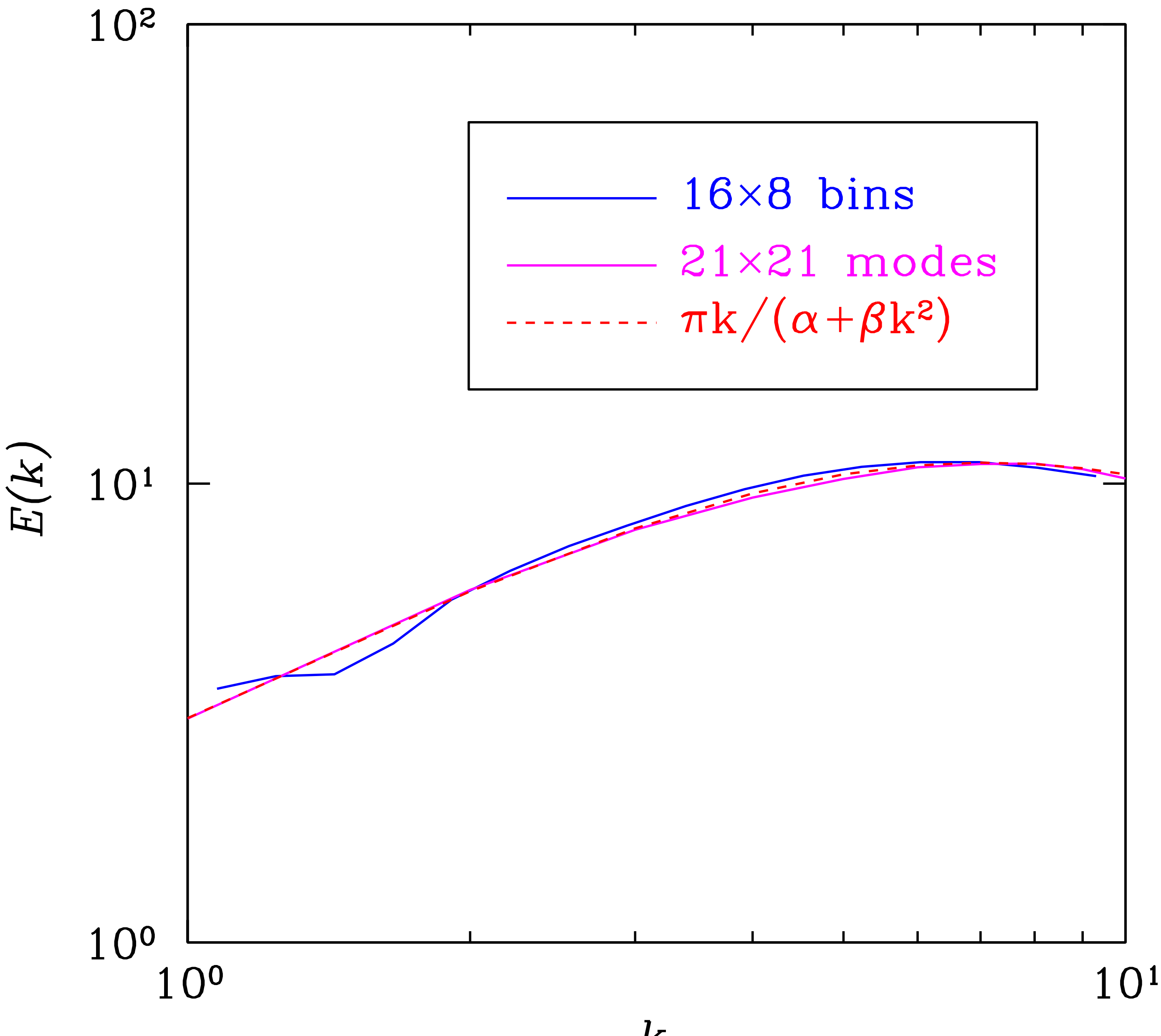
  lead to statistical equipartition of $(\alpha/K^2+\beta)\,|\Omega_{\boldsymbol{K}}|^2\,\Delta_{\boldsymbol{K}}$.

- This is the correct equipartition only for uniform bins. However, for nonuniform bins, a rescaling of time by $\Delta_{\boldsymbol{K}}$:

$$\frac{1}{\Delta_{\boldsymbol{K}}}\frac{\partial\Omega_{\boldsymbol{K}}}{\partial t} + \langle\nu_{\boldsymbol{k}}\rangle_{\boldsymbol{K}}\,\Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}}\Delta_{\boldsymbol{Q}}\,\frac{\left\langle\epsilon_{\boldsymbol{kpq}}\right\rangle_{\boldsymbol{KPQ}}}{Q^2}\,\Omega^*_{\boldsymbol{P}}\Omega^*_{\boldsymbol{Q}}.$$

  yields the correct inviscid equipartition:

$$\left\langle|\Omega_{\boldsymbol{k}}|^2\right\rangle = \frac{1}{\frac{\alpha}{K^2}+\beta}.$$



Relaxation to equipartition

Stiffness Problem

- The rescaling of time does not change the steady-state moment equations.
- It does affect the statistical trajectory of the system and the resulting statistical solution.
- However, the resulting system becomes numerically very stiff.
- Unsolved Problem: given an efficient numerical method for evolving the system of equations

$$\frac{d\boldsymbol{y}}{dt} = \boldsymbol{S}(\boldsymbol{y}),$$

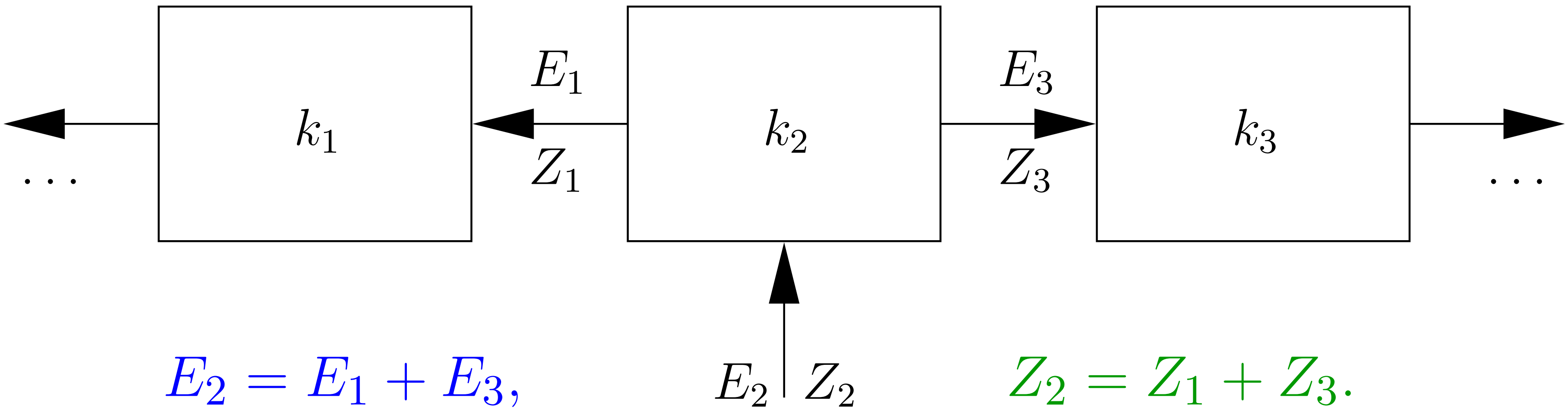
find an efficient numerical method to evolve

$$\frac{d\boldsymbol{y}}{dt} = \boldsymbol{\Lambda}\boldsymbol{S}(\boldsymbol{y}),$$

where $\boldsymbol{\Lambda}$ is a constant real diagonal matrix.

KLB Theory of 2D Turbulence

- Energy $E = \frac{1}{2}\sum_{\boldsymbol{k}} \frac{|\omega_{\boldsymbol{k}}|^2}{k^2}$ and enstrophy $Z = \frac{1}{2}\sum_{\boldsymbol{k}} |\omega_{\boldsymbol{k}}|^2$ are conserved.



$E_2 = E_1 + E_3,$ $\qquad$ $Z_2 = Z_1 + Z_3.$

- [Fjørtoft 1953]: energy cascades to large scales and enstrophy cascades to small scales.
- [Kraichnan 1967], [Leith 1968], and [Batchelor 1969] (KLB):
  $k^{-5/3}$ inverse energy cascade on large scales,
  k^{-3} direct enstrophy cascade on small scales.

2D Enstrophy Cascade

- KLB Theory: Enstrophy transfer rate is independent of k.
- Enstrophy transfer rate is proportional to [Ellison 1962, Kraichnan 1971]

$$\bar{\Pi}_Z(k) \doteq \left[\int_0^k p^2 E(p)\, dp\right]^{1/2} \underbrace{k^3 E(k)}_{f}.$$

Let $f(k) \doteq k^3 E(k)$. Differentiate with respect to k:

$$-2\bar{\Pi}^2 \frac{f'}{f^4} = \frac{1}{k}.$$

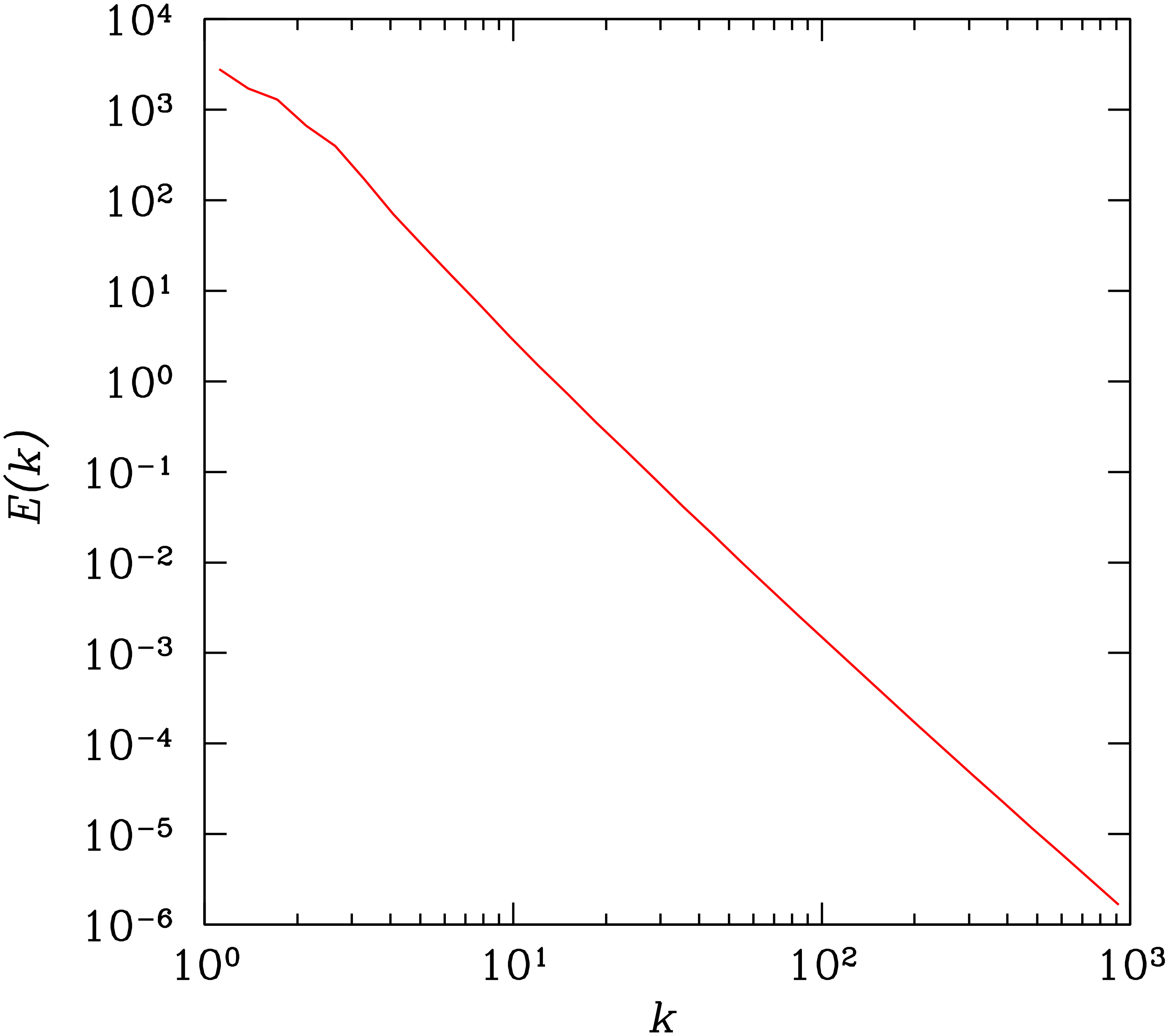
- Let k_1 be the smallest wavenumber in the inertial range.
- Integration between k_1 and k [Bowman 1996] $\Rightarrow$

$$\boxed{E(k) \sim k^{-3}\left[\log\left(\frac{k}{k_1}\right) + \chi_1\right]^{-1/3},} \qquad (k \geq k_1),$$

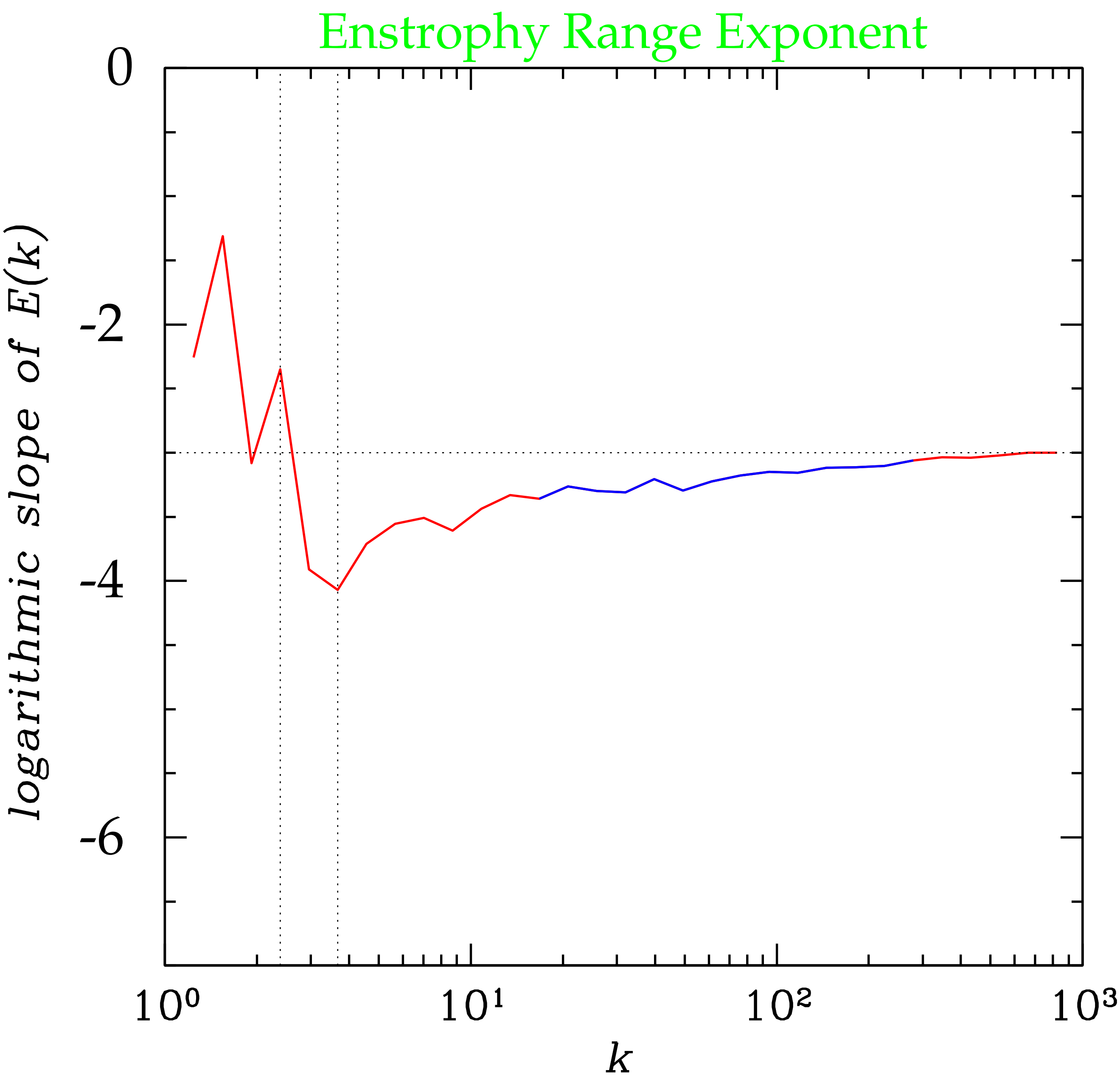
where $\chi_1 \doteq 2\bar{\Pi}_Z^2 k_1^{-9} E^{-3}(k_1)/3$.

- Since $\chi_1 > 0$, there is no divergence at $k = k_1$, in contrast to Kraichnan's result:

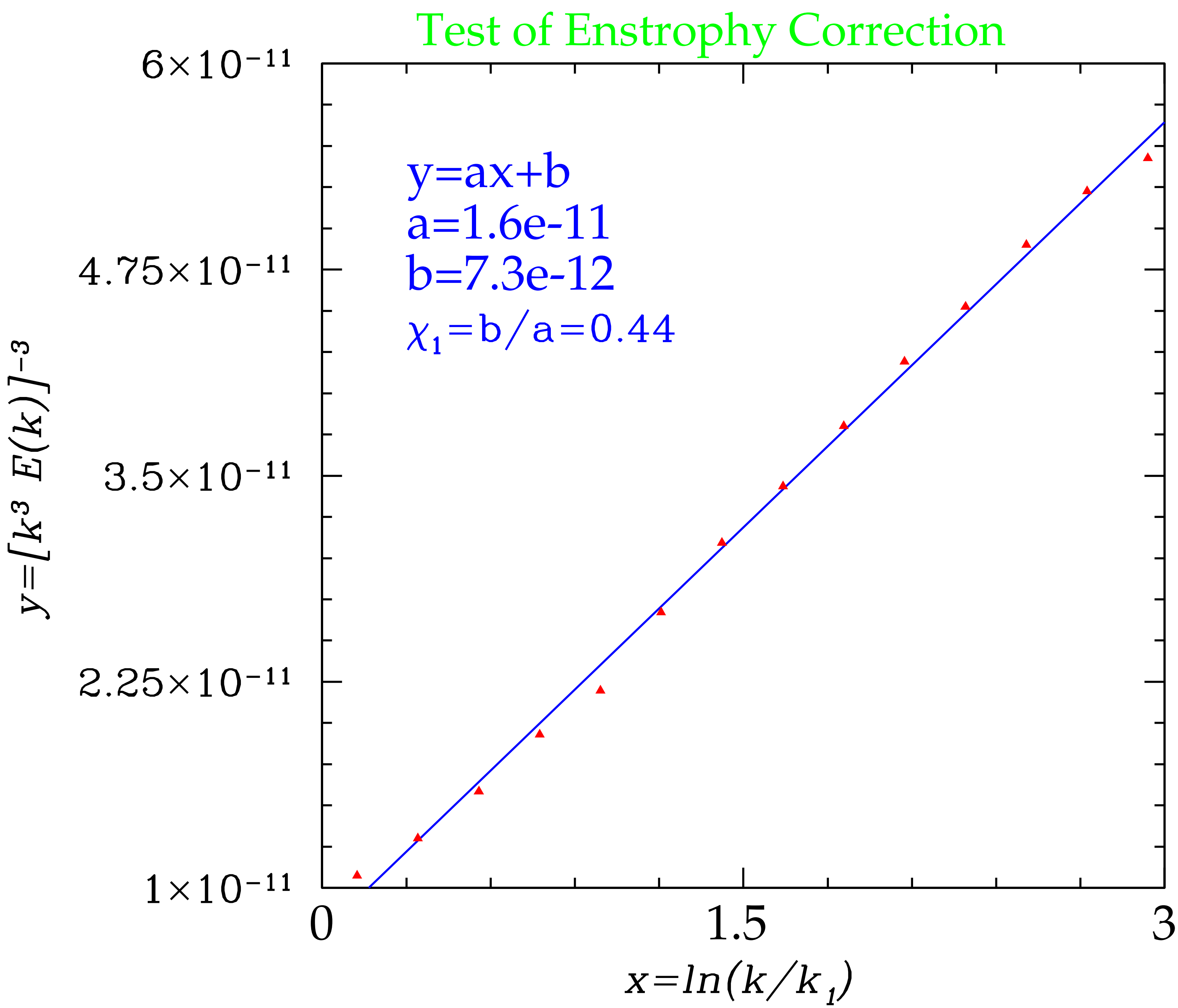
$$E(k) \sim k^{-3}\left[\log\left(\frac{k}{k_1}\right)\right]^{-1/3} \qquad (k \gg k_1).$$



10^4
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
$E(k)$
10^0
10^1
10^2
10^3
k

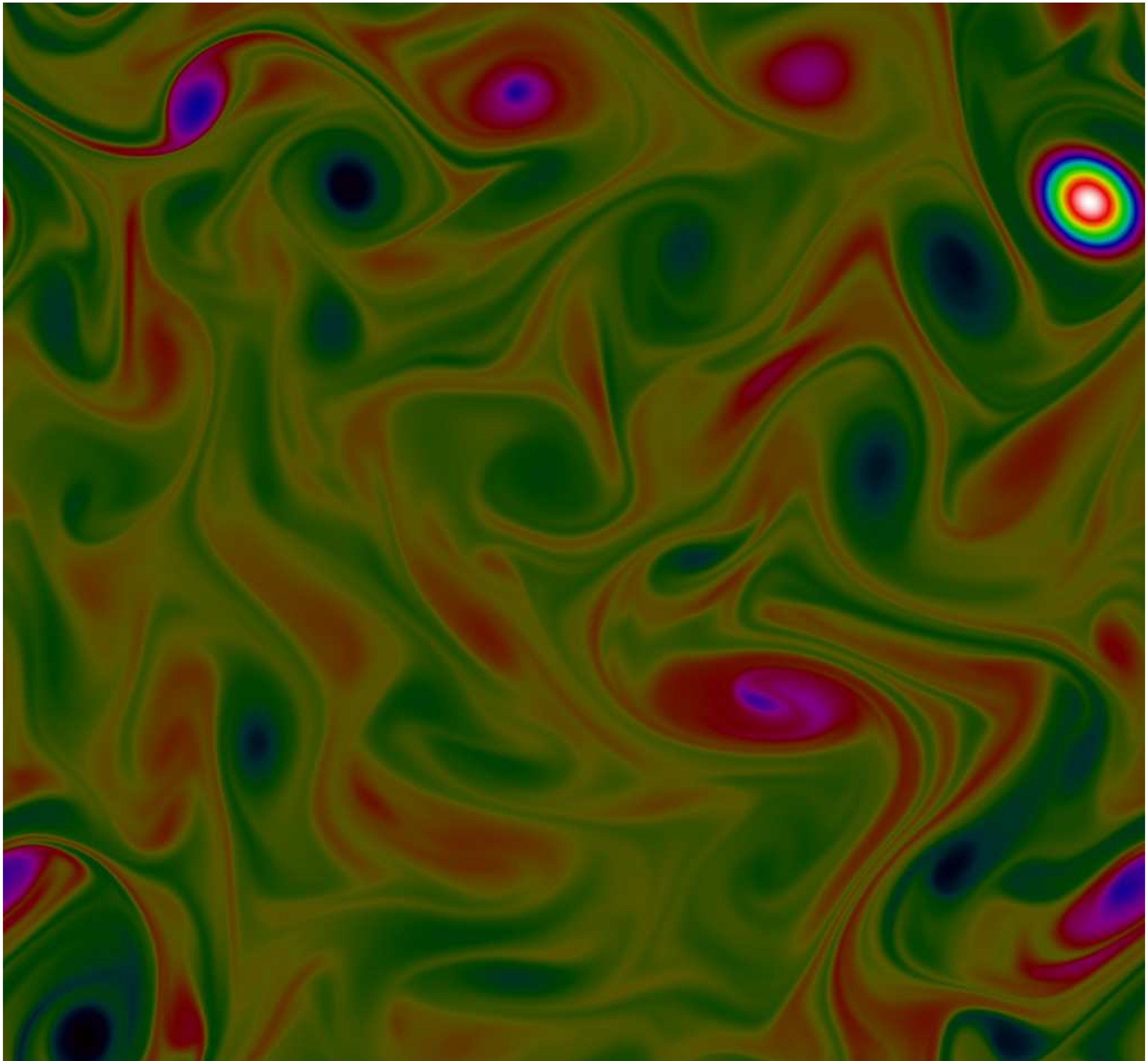


Enstrophy Range Exponent
0
-2
-4
-6
logarithmic slope of $E(k)$
10^0
10^1
10^2
10^3
k



Test of Enstrophy Correction
y=ax+b
a=1.6e-11
b=7.3e-12
χ_1=b/a=0.44
6×10^-11
4.75×10^-11
3.5×10^-11
2.25×10^-11
1×10^-11
0
1.5
3
$x=ln(k/k_1)$
$y=[k^3\ E(k)]^{-3}$

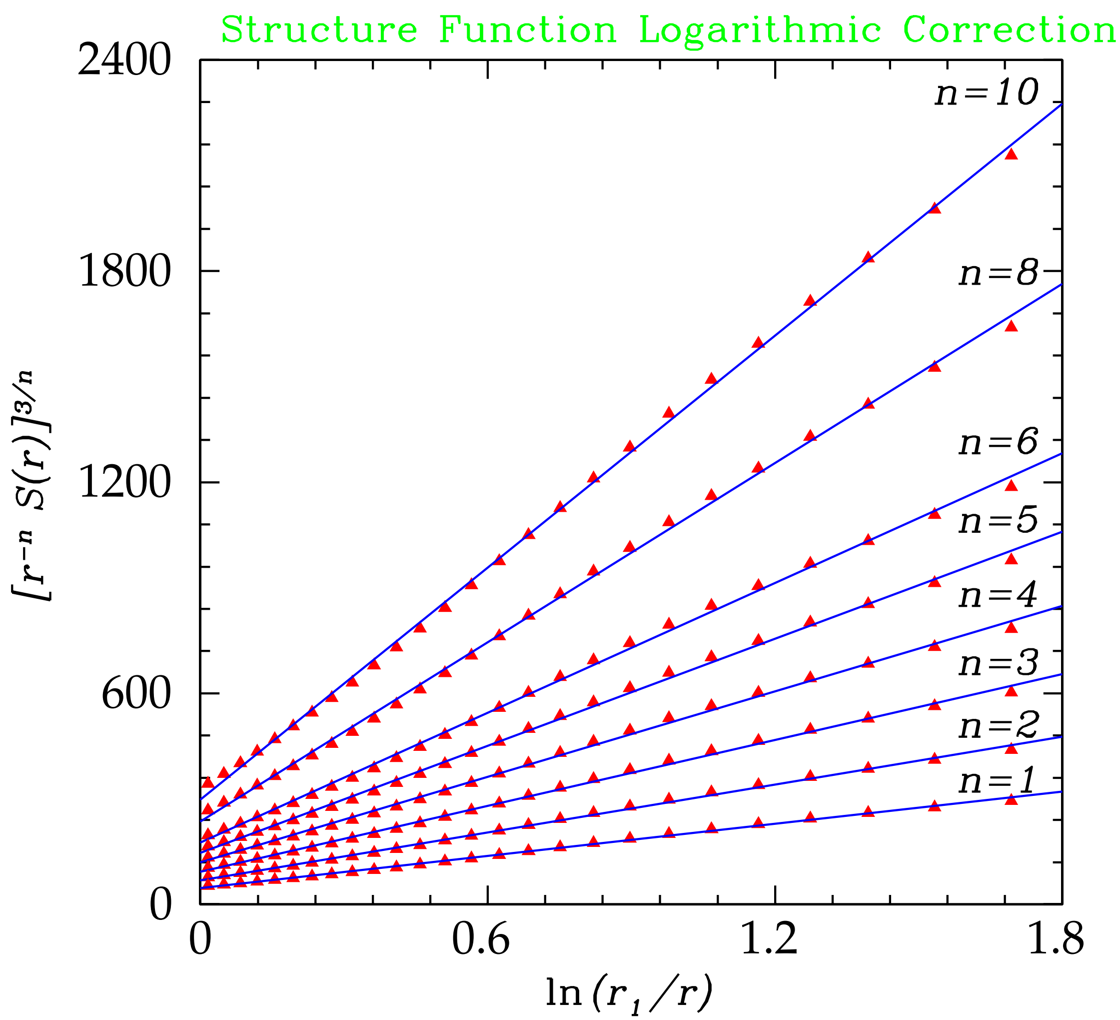
Vorticity Field



Structure functions:

- [Falkovich & Lebedev 1994], [Paret *et al.* 1999]

$$S_n(\boldsymbol{r}) \doteq \overline{|v(\boldsymbol{r}) - v(\boldsymbol{0})|^n} \sim r^n \left[\log\left(\frac{r_1}{r}\right) + \chi'_n\right]^{n/3}.$$



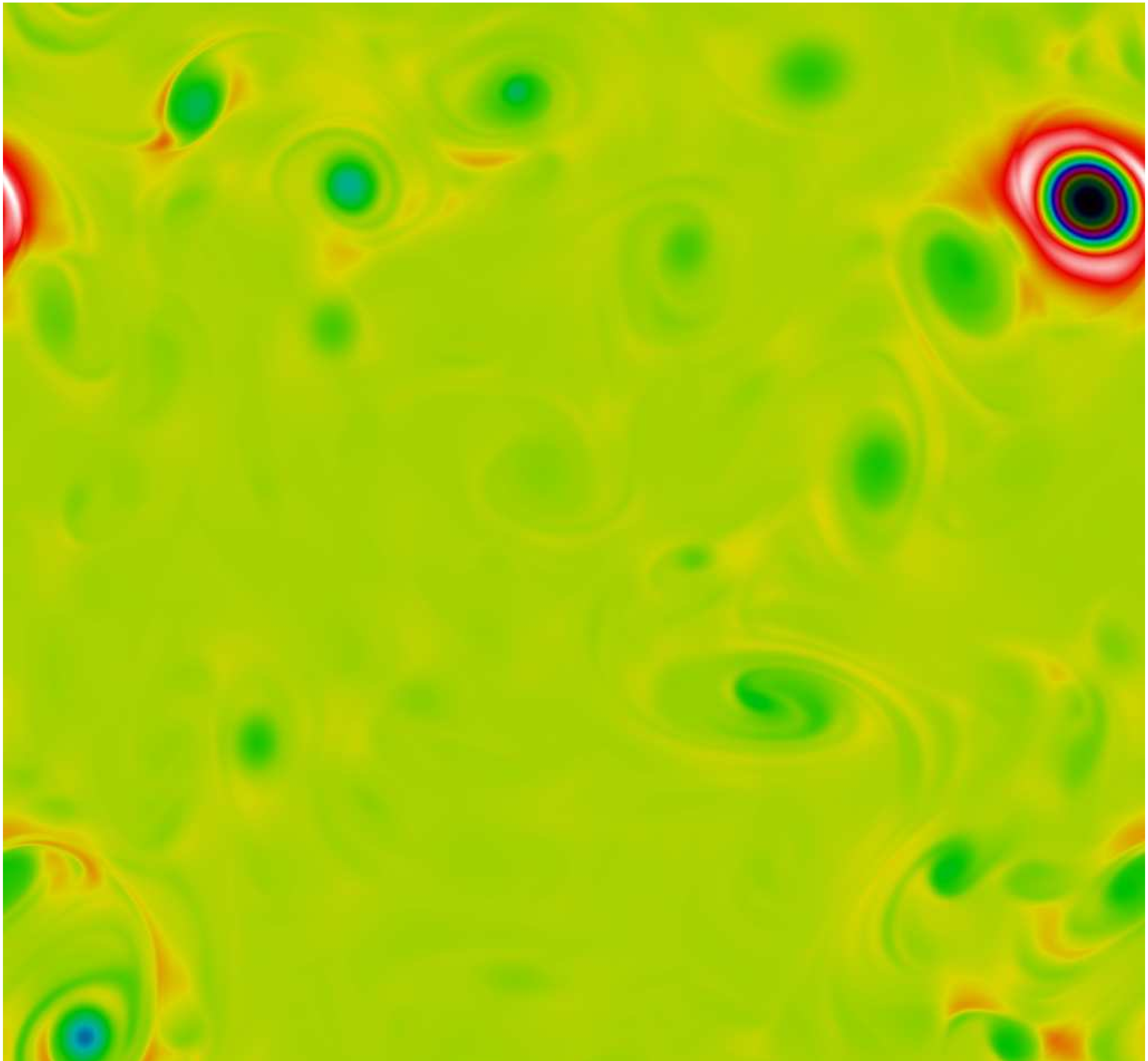
Coherent Structures

- Weiss criterion [Weiss 1991, Pedersen 1995] for coherent structures:

$$Q = \frac{1}{4}\Big(\overbrace{\sigma^2}^{\text{strain}} - \overbrace{\omega^2}^{\text{rotation}} \Big)$$
$$= \psi_{xy}^2 - \psi_{xx}\psi_{yy}$$

- $Q < 0$ (elliptic) $\Rightarrow$ rotation (coherent structures)
- $Q > 0$ (hyperbolic) $\Rightarrow$ strain (deformation)

Weiss Field



Conclusions

- Spectral Reduction affords a dramatic reduction in the number of degrees of freedom that must be explicitly evolved in turbulence simulations.
- One can evolve a turbulent system for thousands of eddy turnover times to obtain energy spectra smooth enough to compare with theory.
- Spectral Reduction has been successfully applied to numerically verify the logarithmically corrected 2D enstrophy law to very high accuracy.
- The high-order structure functions computed by the pseudospectral method and Spectral Reduction are in excellent agreement at small scales, even in the presence coherent structures.
- Spectral Reduction lends numerical support to the theoretical and experimental claim that there are no intermittency corrections in strongly forced 2D enstrophy cascades.

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