On the Global Attractor of 2D Incompressible Turbulence with Random Forcing

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Turbulence

Big whirls have little whirls that feed on their velocity, and little whirls have littler whirls and so on to viscosity... [Richardson 1922]

In 1941, Kolmogorov conjectured that the energy spectrum of 3D incompressible turbulence exhibits a self-similar power-law scaling characterized by a uniform cascade of energy to molecular (viscous) scales:

\[ E(k) = C\epsilon^{2/3}k^{-5/3}. \]
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● Here \( k \) is the Fourier wavenumber and \( E(k) \) is normalized so that \( \int E(k) \, dk \) is the total energy.
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- In 1941, Kolmogorov conjectured that the energy spectrum of 3D incompressible turbulence exhibits a self-similar power-law scaling characterized by a uniform *cascade* of energy to molecular (viscous) scales:

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- Kolmogorov suggested that \( C \) might be a universal constant.
3D Energy Cascade

\[ \log E(k) \]

\[ \log k \]

Forcing Range

Inertial Range

Dissipation Range

\[ k^{-5/3} \]

\[ k_f \]

\[ k_d \]

\[ \log k \]
2D Incompressible Turbulence

• In 2D, where \( u \) maps a plane normal to \( \hat{z} \) to \( R^2 \), the vorticity vector \( \omega = \nabla \times u \) is always perpendicular to \( u \).
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- Navier–Stokes equation for the scalar vorticity \( \omega = \mathbf{z} \cdot \nabla \times u \):

\[
\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \nu \nabla^2 \omega + f.
\]
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\frac{\partial \mathbf{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{\omega} = \nu \nabla^2 \mathbf{\omega} + \mathbf{f}.
\]

• The incompressibility condition \( \nabla \cdot \mathbf{u} = 0 \) can be exploited to find \( \mathbf{u} \) in terms of \( \mathbf{\omega} \):

\[
\nabla \omega \times \hat{z} = \nabla \times \hat{z} \mathbf{\omega} = \nabla \times (\nabla \times \mathbf{u}) = \nabla (\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u} = -\nabla^2 \mathbf{u}.
\]
2D Incompressible Turbulence

• In 2D, where $\mathbf{u}$ maps a plane normal to $\hat{z}$ to $R^2$, the vorticity vector $\omega = \nabla \times \mathbf{u}$ is always perpendicular to $\mathbf{u}$.

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$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nu \nabla^2 \omega + f.$$

• The incompressibility condition $\nabla \cdot \mathbf{u} = 0$ can be exploited to find $\mathbf{u}$ in terms of $\omega$:

$$\nabla \omega \times \hat{z} = \nabla \times \hat{z} \omega = \nabla \times (\nabla \times \mathbf{u}) = \nabla (\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u} = -\nabla^2 \mathbf{u}.$$

• Thus $\mathbf{u} = \hat{z} \times \nabla \nabla^{-2} \omega$. In Fourier space:

$$\frac{d \omega_k}{dt} = S_k - \nu k^2 \omega_k + f_k,$$

where $S_k = \sum_q \frac{\hat{z} \times \mathbf{q} \cdot \mathbf{k}}{q^2} \overline{\omega_q \omega_{-\mathbf{k} - \mathbf{q}}} = \sum_{p, q} \frac{\epsilon_{kpq}}{q^2} \overline{\omega_p \omega_q}$.
Here $\epsilon_{kpq} \equiv \hat{z} \cdot p \times q \delta_{k+p+q}$ is antisymmetric under permutation of any two indices.

$$\frac{d\omega_k}{dt} + \nu k^2 \omega_k = \sum_p \sum_q \frac{\epsilon_{kpq}}{q^2} \omega_p \omega_q + f_k,$$

- When $\nu = f_k = 0$:

  enstrophy $Z = \frac{1}{2} \sum_k |\omega_k|^2$ and energy $E = \frac{1}{2} \sum_k \frac{|\omega_k|^2}{k^2}$ are conserved:

  \[
  \frac{\epsilon_{kpq}}{q^2} \quad \text{antisymmetric in} \quad k \leftrightarrow p,
  \]

  \[
  \frac{1}{k^2} \frac{\epsilon_{kpq}}{q^2} \quad \text{antisymmetric in} \quad k \leftrightarrow q.
  \]
Fjørtoft Dual Cascade Scenario

\[ E_2 = E_1 + E_3, \quad Z_2 = Z_1 + Z_3, \quad Z_i \approx k_i^2 E_i. \]

- When \( k_1 = k, \ k_2 = 2k, \) and \( k_3 = 4k: \)

\[ E_1 \approx \frac{4}{5} E_2, \quad Z_1 \approx \frac{1}{5} Z_2, \quad E_3 \approx \frac{1}{5} E_2, \quad Z_3 \approx \frac{4}{5} Z_2. \]
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- Fjørtoft [1953]: energy cascades to large scales and enstrophy cascades to small scales.
2D Energy Cascade

\[ \log E(k) \]

Forcing Range

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Dissipation Range

\[ k^{-3} \]

\[ k^{-5/3} \]
2D Turbulence: Mathematical Formulation

Consider the Navier–Stokes equations for 2D incompressible homogeneous isotropic turbulence with density $\rho = 1$:

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \nabla^2 \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \mathbf{F},$$

$$\nabla \cdot \mathbf{u} = 0,$$

$$\int_{\Omega} \mathbf{u} \, d\mathbf{x} = 0, \quad \int_{\Omega} \mathbf{F} \, d\mathbf{x} = 0,$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}),$$

with $\Omega = [0, 2\pi] \times [0, 2\pi]$ and periodic boundary conditions on $\partial \Omega$. 
2D Turbulence: Mathematical Formulation

- Consider the Navier–Stokes equations for 2D incompressible homogeneous isotropic turbulence with density $\rho = 1$:

$$
\begin{align*}
\frac{\partial u}{\partial t} - \nu \nabla^2 u + u \cdot \nabla u + \nabla P &= F, \\
\nabla \cdot u &= 0, \\
\int_{\Omega} u \, dx &= 0, \quad \int_{\Omega} F \, dx = 0, \\
u_0(x),
\end{align*}
$$

with $\Omega = [0, 2\pi] \times [0, 2\pi]$ and periodic boundary conditions on $\partial \Omega$.

- Introduce the Hilbert space

$$
H(\Omega) \equiv \text{cl} \left\{ u \in (C^2(\Omega) \cap L^2(\Omega))^2 \mid \nabla \cdot u = 0, \quad \int_{\Omega} u \, dx = 0 \right\}.
$$

with inner product $(u, v) = \int_{\Omega} u(x, t) \cdot v(x, t) \, dx$ and $L^2$ norm $|u| = (u, u)^{1/2}$. 

For \( u \in H(\Omega) \), the Navier–Stokes equations can be expressed:

\[
\frac{d u}{d t} - \nu \nabla^2 u + u \cdot \nabla u + \nabla P = F.
\]
• For $u \in H(\Omega)$, the Navier–Stokes equations can be expressed:

$$\frac{d}{dt}u - \nu \nabla^2 u + u \cdot \nabla u + \nabla P = F.$$ 

• Introduce $A := -\mathcal{P}(\nabla^2)$, $f := \mathcal{P}(F)$, and the bilinear map

$$\mathcal{B}(u, u) := \mathcal{P}(u \cdot \nabla u + \nabla P),$$

where $\mathcal{P}$ is the Helmholtz–Leray projection operator from $(L^2(\Omega))^2$ to $H(\Omega)$:

$$\mathcal{P}(v) := v - \nabla \nabla^{-2} \nabla \cdot v, \quad \forall v \in (L^2(\Omega))^2.$$
• For \( \mathbf{u} \in H(\Omega) \), the Navier–Stokes equations can be expressed:

\[
\frac{d\mathbf{u}}{dt} - \nu \nabla^2 \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = F.
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• Introduce \( A \doteq -\mathcal{P}(\nabla^2) \), \( f \doteq \mathcal{P}(F) \), and the bilinear map

\[
\mathcal{B}(\mathbf{u}, \mathbf{u}) \doteq \mathcal{P}(\mathbf{u} \cdot \nabla \mathbf{u} + \nabla P),
\]

where \( \mathcal{P} \) is the Helmholtz–Leray projection operator from \( (L^2(\Omega))^2 \) to \( H(\Omega) \):

\[
\mathcal{P}(\mathbf{v}) \doteq \mathbf{v} - \nabla \nabla^{-2} \nabla \cdot \mathbf{v}, \quad \forall \mathbf{v} \in (L^2(\Omega))^2.
\]

• The dynamical system can then be compactly written:

\[
\frac{d\mathbf{u}}{dt} + \nu A\mathbf{u} + \mathcal{B}(\mathbf{u}, \mathbf{u}) = f.
\]
Stokes Operator $A$

- The operator $A = \mathcal{P}(-\nabla^2)$ is positive semi-definite and self-adjoint, with a compact inverse.
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- On the periodic domain $\Omega = [0, 2\pi] \times [0, 2\pi]$, the eigenvalues of $A$ are

  $$\lambda = k \cdot k, \quad k \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\}.$$
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  $$\lambda = k \cdot k, \quad k \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\}.$$ 

- The eigenvalues of $A$ can be arranged as

  $$0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \quad \lambda_0 = 1$$

and its eigenvectors $\mathbf{w}_i, \ i \in \mathbb{N}_0$, form an orthonormal basis for the Hilbert space $H$, upon which we can define any quotient power of $A$:

$$A^\alpha \mathbf{w}_j = \lambda_j^\alpha \mathbf{w}_j, \quad \alpha \in \mathbb{R}, \ j \in \mathbb{N}_0.$$
Subspace of Finite Enstrophy

- We define the subspace of $H$ consisting of solutions with finite enstrophy:

$$V \doteq \left\{ u \in H \mid \sum_{j=0}^{\infty} \lambda_j(u, w_j)^2 < \infty \right\}.$$
Subspace of Finite Enstrophy

• We define the subspace of $H$ consisting of solutions with finite enstrophy:

$$V = \left\{ \mathbf{u} \in H \mid \sum_{j=0}^{\infty} \lambda_j (\mathbf{u}, \mathbf{w}_j)^2 < \infty \right\}.$$ 

• Another suitable norm for elements $\mathbf{u} \in V$ is

$$||\mathbf{u}|| = \bigg| A^{1/2} \mathbf{u} \bigg| = \left( \int_{\Omega} \sum_{i=1}^{2} \frac{\partial \mathbf{u}}{\partial x_i} \cdot \frac{\partial \mathbf{u}}{\partial x_i} \right)^{1/2} = \left( \sum_{j=0}^{\infty} \lambda_j (\mathbf{u}, \mathbf{w}_j)^2 \right)^{1/2}.$$
Properties of the Bilinear Map

- We will make use of the antisymmetry

\[(\mathcal{B}(u, v), w) = -(\mathcal{B}(u, w), v).\]
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\[(\mathcal{B}(u, v), w) = -(\mathcal{B}(u, w), v).\]

• In 2D, we also have orthogonality:

\[(\mathcal{B}(u, u), Au) = 0\]

and the strong form of enstrophy invariance:

\[(\mathcal{B}(Au, v), u) = (\mathcal{B}(u, v), Av).\]
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and the strong form of enstrophy invariance:

\[(\mathcal{B}(Av, v), u) = (\mathcal{B}(u, v), Av).\]

• In 2D the above properties imply the symmetry

\[(\mathcal{B}(Au, u), u) + (\mathcal{B}(v, Av), u) + (\mathcal{B}(v, v), Av) = 0.\]
Dynamical Behaviour

- Our starting point is the incompressible 2D Navier–Stokes equation with periodic boundary conditions:

\[
\frac{du}{dt} + \nu Au + B(u, u) = f, \quad u \in H.
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• Take the inner product with \( u \) (respectively \( Au \)):

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\frac{1}{2} \frac{d}{dt} |u(t)|^2 + \nu |u(t)|^2 = (f, u(t)),
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Dynamical Behaviour

• Our starting point is the incompressible 2D Navier–Stokes equation with periodic boundary conditions:

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\frac{d\mathbf{u}}{dt} + \nu A\mathbf{u} + B(\mathbf{u}, \mathbf{u}) = \mathbf{f}, \quad \mathbf{u} \in H.
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• Take the inner product with \( \mathbf{u} \) (respectively \( A\mathbf{u} \)):

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\]

• The Cauchy–Schwarz and Poincaré inequalities yield

\[
(f, \mathbf{u}(t)) \leq |\mathbf{f}||\mathbf{u}(t)| \quad \text{and} \quad |\mathbf{u}(t)| \leq ||\mathbf{u}(t)||.
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(f, u(t)) \leq |f||u(t)| \quad \text{and} \quad |u(t)| \leq \|u(t)\|.
\]

- Since the existence and uniqueness for solutions to the 2D Navier–Stokes equation has been proven, a global attractor can be defined [Ladyzhenskaya 1975], [Foias & Temam 1979].
Dynamical Behaviour: Constant Forcing

- If the force $f$ is constant with respect to time, a Gronwall inequality can be exploited:

$$|u(t)|^2 \leq e^{-\nu t}|u(0)|^2 + (1 - e^{-\nu t}) \left(\frac{|f|}{\nu}\right)^2.$$
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• Defining a nondimensional Grashof number $G = \frac{|f|}{\nu^2}$, the above inequality can be simplified to

$$|u(t)|^2 \leq e^{-\nu t}|u(0)|^2 + (1 - e^{-\nu t})\nu^2 G^2.$$
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• Defining a nondimensional Grashof number $G = \frac{|f|}{\nu^2}$, the above inequality can be simplified to

$$|u(t)|^2 \leq e^{-\nu t} |u(0)|^2 + (1 - e^{-\nu t}) \nu^2 G^2.$$  

• Similarly,

$$||u(t)||^2 \leq e^{-\nu t} ||u(0)||^2 + (1 - e^{-\nu t}) \nu^2 G^2.$$  

• Being on the attractor thus requires

$$|u| \leq \nu G \quad \text{and} \quad ||u|| \leq \nu G.$$
Attractor Set $\mathcal{A}$

- Let $S$ be the solution operator:

$$S(t)u_0 = u(t), \quad u_0 = u(0),$$

where $u(t)$ is the unique solution of the Navier–Stokes equations.
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The closed ball $\mathfrak{B}$ of radius $\nu G$ in the space $V$ is a bounded absorbing set in $H$. 
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where $u(t)$ is the unique solution of the Navier–Stokes equations.

- The closed ball $\mathcal{B}$ of radius $\nu G$ in the space $V$ is a bounded absorbing set in $H$.

- That is, for any bounded set $\mathcal{B}'$ there exists a time $t_0$ such that

$$t_0 = t_0(\mathcal{B}'), \quad \text{and} \quad S(t)\mathcal{B}' \subset \mathcal{B}, \quad \forall t \geq t_0.$$

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$$t_0 = t_0(\mathcal{B}'), \quad \text{and} \quad S(t)\mathcal{B}' \subset \mathcal{B}, \quad \forall t \geq t_0.$$

- We can then construct the global attractor:

$$\mathcal{A} = \bigcap_{t \geq 0} S(t)\mathcal{B},$$

so $\mathcal{A}$ is the largest bounded, invariant set such that $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$. 
A trivial lower bound is provided by the Poincaré inequality:

\[ |u|^2 \leq ||u||^2 \implies E \leq Z. \]
Z–E Plane Bounds: Constant Forcing

• A trivial lower bound is provided by the Poincaré inequality:

\[ |u|^2 \leq \|u\|^2 \Rightarrow E \leq Z. \]

• An upper bound is given by

**Theorem 1 (Dascaliuc, Foias, and Jolly [2005])**  
*For all \( u \in A \),

\[ \|u\|^2 \leq \frac{|f|}{\nu} |u|. \]
Z–E Plane Bounds: Constant Forcing

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• An upper bound is given by

**Theorem 2 (Dascaliuc, Foias, and Jolly [2005])**

For all \( u \in A \),

\[ ||u||^2 \leq \frac{|f|}{\nu}|u|. \]

• That is,

\[ Z \leq \nu G \sqrt{E}. \]
$Z - E$ Plane Bounds: Constant Forcing

\[
\frac{2Z}{\nu^2 G^2} \quad 0 \quad 1
\]

\[
\frac{2E}{\nu^2 G^2} \quad 0 \quad 1
\]

A in here
Extended Norm: Random Forcing

For a random variable $\alpha$, with probability density function $P$, define the ensemble average

$$\langle \alpha \rangle = \int_{-\infty}^{\infty} \alpha \left( \frac{dP}{d\zeta} \right) d\zeta.$$
Extended Norm: Random Forcing

- For a random variable $\alpha$, with probability density function $P$, define the ensemble average

$$\langle \alpha \rangle = \int_{-\infty}^{\infty} \alpha \left( \frac{dP}{d\zeta} \right) d\zeta.$$

- The extended inner product is

$$(u, v)_{\tilde{\omega}} = \int_{\Omega} \langle u \cdot v \rangle \, dx = \int_{\Omega} \left( \int_{-\infty}^{\infty} u \cdot v \frac{dP}{d\zeta} \, d\zeta \right) \, dx,$$

with norm

$$|f|_{\tilde{\omega}} = \left( \int_{\Omega} \langle |f|^2 \rangle \, dx \right)^{1/2}.$$
Dynamical Behaviour: Random Forcing

- Energy balance:

\[
\frac{1}{2} \frac{d}{dt} |u|^2 + \nu(Au, u) + (B(u, u), u) = (f, u) = \epsilon,
\]

where $\epsilon$ is the rate of energy injection.
Dynamical Behaviour: Random Forcing

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\]

where \( \epsilon \) is the rate of energy injection.

• From the energy conservation identity \((B(u, u), u) = 0\),

\[
\frac{1}{2} \frac{d}{dt} |u|^2 + \nu ||u||^2 = \epsilon.
\]
Dynamical Behaviour: Random Forcing

- Energy balance:

\[
\frac{1}{2} \frac{d}{dt} |\mathbf{u}|^2 + \nu (A\mathbf{u}, \mathbf{u}) + (\mathcal{B}(\mathbf{u}, \mathbf{u}), \mathbf{u}) = (\mathbf{f}, \mathbf{u}) \dot{=} \epsilon,
\]

where \(\epsilon\) is the rate of energy injection.

- From the energy conservation identity \((\mathcal{B}(\mathbf{u}, \mathbf{u}), \mathbf{u}) = 0\),

\[
\frac{1}{2} \frac{d}{dt} |\mathbf{u}|^2 + \nu |\mathbf{u}|^2 = \epsilon.
\]

- The Poincaré inequality \(||\mathbf{u}|| \geq |\mathbf{u}|\) leads to

\[
\frac{1}{2} \frac{d}{dt} |\mathbf{u}|^2 \leq \epsilon - \nu |\mathbf{u}|^2,
\]

which implies that 

\[
|\mathbf{u}(t)|^2 \leq e^{-2\nu t} |\mathbf{u}(0)|^2 + \left(1 - \frac{e^{-2\nu t}}{\nu}\right) \epsilon.
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Dynamical Behaviour: Random Forcing

- Energy balance:
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  \frac{1}{2} \frac{d}{dt} |u|^2 + \nu (Au, u) + (B(u, u), u) = (f, u) \dot{=} \epsilon,
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  where \( \epsilon \) is the rate of energy injection.

- From the energy conservation identity \((B(u, u), u) = 0\),
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- The Poincaré inequality \(||u|| \geq |u|\) leads to
  \[
  \frac{1}{2} \frac{d}{dt} |u|^2 \leq \epsilon - \nu |u|^2,
  \]
  which implies that \(|u(t)|^2 \leq e^{-2\nu t} |u(0)|^2 + \left( \frac{1 - e^{-2\nu t}}{\nu} \right) \epsilon.\]

- So for every \( u \in A \), we expect \(|u(t)|^2 \leq \epsilon / \nu.\)
From $|u(t)| \leq \sqrt{\epsilon / \nu}$ we then obtain a lower bound for $|f|$: 

$$\sqrt{\nu \epsilon} \leq \frac{\epsilon}{|u|} = \frac{(f, u)}{|u|} \leq \frac{|f||u|}{|u|} = |f|.$$
• From $|u(t)| \leq \sqrt{\epsilon/\nu}$ we then obtain a lower bound for $|f|$:

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• It is convenient to use this lower bound for $|f|$ to define a lower bound for the Grashof number $G = \frac{|f|}{\nu^2}$, which we use as the normalization $\tilde{G}$ for random forcing:

$$\tilde{G} = \sqrt{\frac{\epsilon}{\nu^3}}.$$
• From $|\mathbf{u}(t)| \leq \sqrt{\epsilon/\nu}$ we then obtain a lower bound for $|\mathbf{f}|$:

\[
\sqrt{\nu \epsilon} \leq \frac{\epsilon}{|\mathbf{u}|} = \frac{(\mathbf{f}, \mathbf{u})}{|\mathbf{u}|} \leq \frac{||\mathbf{f}||_{\mathbf{u}}}{|\mathbf{u}|} = |\mathbf{f}|.
\]

• It is convenient to use this lower bound for $|\mathbf{f}|$ to define a lower bound for the Grashof number $G = |\mathbf{f}|/\nu^2$, which we use as the normalization $\tilde{G}$ for random forcing:

\[
\tilde{G} = \sqrt{\frac{\epsilon}{\nu^3}}.
\]

• We recently proved the following theorem (submitted to JDE):

**Theorem 3 (Emami & Bowman [2017])** For all $\mathbf{u} \in \mathcal{A}$ with energy injection rate $\epsilon$,

\[
||\mathbf{u}||^2 \leq \sqrt{\frac{\epsilon}{\nu}} |\mathbf{u}|.
\]
• From $|u(t)| \leq \sqrt{\epsilon/\nu}$ we then obtain a lower bound for $|f|$

$$\sqrt{\nu \epsilon} \leq \frac{\epsilon}{|u|} = \frac{(f, u)}{|u|} \leq \frac{|f||u|}{|u|} = |f|.$$  

• It is convenient to use this lower bound for $|f|$ to define a lower bound for the Grashof number $G = |f|/\nu^2$, which we use as the normalization $\tilde{G}$ for random forcing:

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• We recently proved the following theorem (submitted to JDE): 

**Theorem 4** (Emami & Bowman [2017]) *For all* $u \in A$ *with energy injection rate* $\epsilon$,

$$||u||^2 \leq \sqrt{\frac{\epsilon}{\nu}} |u|.$$  

• This leads to the same form as for a constant force: $Z \leq \nu \tilde{G} \sqrt{E}$. 


$Z - E$ Plane Bounds: Random Forcing

$\frac{2Z}{\nu^2 G^2}$

$\frac{2E}{\nu^2 G^2}$

A in here
DNS code

- We have released a highly optimized 2D pseudospectral code in C++: https://github.com/dealias/dns.
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• It uses the formulation proposed by Basdevant [1983] to reduce the number of FFTs required for 2D (3D) incompressible turbulence to 4 (8).

• We also include simplified 2D (146 lines) and 3D (287 lines) versions called ProtoDNS for educational purposes: https://github.com/dealias/dns/tree/master/protodns.
Dynamic Moment Averaging

- Advantageous to precompute time-integrated moments like

\[ M_n(t) = \int_0^t |\omega_k(\tau)|^n \, d\tau. \]
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along with the vorticity \( \omega_k \) itself, using *the same* temporal discretization.
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along with the vorticity \( \omega_k \) itself, using the same temporal discretization.

- These evolved quantities \( M_n \) can be used to extract accurate statistical averages during the post-processing phase, once the saturation time \( t_1 \) has been determined by the user:

\[ \int_{t_1}^{t_2} |\omega_k|^n(\tau) d\tau = M_n(t_2) - M_n(t_1). \]
Enstrophy Balance

\[ \frac{\partial \omega_k}{\partial t} + \nu k^2 \omega_k = S_k + f_k, \]

- Multiply by \( \omega_k^* \) and integrate over wavenumber angle \( \Rightarrow \) enstrophy spectrum \( Z(k) \) evolves as:

\[ \frac{\partial}{\partial t} Z(k) + 2\nu k^2 Z(k) = 2T(k) + G(k), \]

where \( T(k) \) and \( G(k) \) are the corresponding angular averages of \( \text{Re} \langle S_k \omega_k^* \rangle \) and \( \text{Re} \langle f_k \omega_k^* \rangle \).
Nonlinear Enstrophy Transfer Function

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Let

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\Pi(k) = 2 \int_k^{\infty} T(p) \, dp
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represent the nonlinear transfer of enstrophy into \([k, \infty)\).
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• Let

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represent the nonlinear transfer of enstrophy into \([k, \infty)\).

• Integrate from \(k\) to \(\infty\):

\[
\frac{d}{dt} \int_{k}^{\infty} Z(p) \, dp = \Pi(k) - \epsilon_Z(k),
\]

where \(\epsilon_Z(k) \doteq 2\nu \int_{k}^{\infty} p^2 Z(p) \, dp - \int_{k}^{\infty} G(p) \, dp\) is the total enstrophy transfer, via dissipation and forcing, out of wavenumbers higher than \(k\).
A positive (negative) value for $\Pi(k)$ represents a flow of enstrophy to wavenumbers higher (lower) than $k$. 
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• When $\nu = 0$ and $f_k = 0$:

$$0 = \frac{d}{dt} \int_0^\infty Z(p) \, dp = 2 \int_0^\infty T(p) \, dp,$$

so that

$$\Pi(k) = 2 \int_k^\infty T(p) \, dp = -2 \int_0^k T(p) \, dp.$$
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• This provides an excellent numerical diagnostic for determining the saturation time $t_1$. 
Vorticity Field with Hypoviscosity
Energy Spectrum with Hypoviscosity
Bounds in the $Z-E$ plane for random forcing.
Energy Transfer with Hypoviscosity

![Graph showing cumulative enstrophy transfer vs. k]
Vorticity Field without Hypoviscosity
Energy Spectrum without Hypoviscosity

\[ E(k) \]

\[ 10^{-11}, 10^{-9}, 10^{-7}, 10^{-5}, 10^{-3}, 10^{-1}, 10^1 \]

\[ k \]

\[ 10^0, 10^1, 10^2 \]
Bounds in the $Z-E$ plane for random forcing.
Energy Transfer without Hypoviscosity

Cumulative entropy transfer

\[ k \]

\[ \Pi \]

\[ \eta \]
Special Case: White-Noise Forcing

- The Fourier transform of an isotropic Gaussian white-noise solenoidal force $f$ has the form

$$f_k(t) = F_k \left(1 - \frac{k k}{k^2}\right) \cdot \xi_k(t), \quad k \cdot f_k = 0,$$

where $F_k$ is a real number and $\xi_k(t)$ is a unit central real Gaussian random 2D vector that satisfies

$$\langle \xi_k(t) \xi_k'(t') \rangle = \delta_{kk'} 1 \delta(t - t').$$
Special Case: White-Noise Forcing

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• This implies

$$\langle \mathbf{f}_k(t) \cdot \mathbf{f}_{k'}(t') \rangle = F_k^2 \delta_{kk'} \delta(t - t').$$
Special Case: White-Noise Forcing

- To prescribe the forcing amplitude $F_k$ in terms of $\epsilon$:

**Theorem 5 ([Novikov 1964])** If $f(\mathbf{x}, t)$ is a Gaussian process, and $u$ is a functional of $f$, then

$$
\langle f(\mathbf{x}, t) u(f) \rangle = \int \int \langle f(\mathbf{x}, t) f(\mathbf{x}', t') \rangle \left\langle \frac{\delta u(\mathbf{x}, t)}{\delta f(\mathbf{x}', t')} \right\rangle d\mathbf{x}' dt'.
$$
Special Case: White-Noise Forcing

- To prescribe the forcing amplitude $F_k$ in terms of $\epsilon$:

**Theorem 6 (Novikov [1964])** If $f(\mathbf{x}, t)$ is a Gaussian process, and $u$ is a functional of $f$, then

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- For white-noise forcing:

$$\epsilon = \text{Re} \sum_k \langle \mathbf{f}_k(t) \cdot \mathbf{u}_k(t) \rangle = \text{Re} \sum_{k,k'} \int \langle \mathbf{f}_k(t) \mathbf{f}_{k'}(t') \rangle : \left\langle \frac{\delta \mathbf{u}_k(t)}{\delta \mathbf{f}_{k'}(t')} \right\rangle dt'$$

$$= \sum_k F_k^2 \left( 1 - \frac{kk}{k^2} \right) : \left( 1 - \frac{kk}{k^2} \right) H(0)$$

$$= \frac{1}{2} \sum_k F_k^2,$$

on noting that $H(0) = 1/2$. 
White-Noise Forcing: Implementation

- At the end of each time-step, we implement the contribution of white noise forcing with the discretization

\[ \omega_{k,n+1} = \omega_{k,n} + \sqrt{2\tau} \eta_k \xi, \]

where \( \xi \) is a unit complex Gaussian random number with \( \langle \xi \rangle = 0 \) and \( \langle |\xi|^2 \rangle = 1 \).
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where \( \xi \) is a unit complex Gaussian random number with \( \langle \xi \rangle = 0 \) and \( \langle |\xi|^2 \rangle = 1 \).

• This yields the mean enstrophy injection

\[ \frac{\langle |\omega_{k,n+1}|^2 - |\omega_{k,n}|^2 \rangle}{2\tau} = \eta_k. \]
3D Basdevant Formulation: 8 FFTs

Using incompressibility, the 3D momentum equation can be written in terms of the symmetric tensor $D_{ij} = u_i u_j$:

$$\frac{\partial u_i}{\partial t} + \frac{\partial D_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + F_i.$$
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- Naive implementation: 3 backward FFTs to compute the velocity components from their spectral representations, 6 forward FFTs of the independent components of $D_{ij}$. 
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• Basdevant [1983]: avoid one FFT by subtracting the divergence of the symmetric matrix $S_{ij} = \delta_{ij} \text{tr} D/3$ from both sides:

$$\frac{\partial u_i}{\partial t} + \frac{\partial (D_{ij} - S_{ij})}{\partial x_j} = -\frac{\partial (p\delta_{ij} + S_{ij})}{\partial x_j} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + F_i.$$
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• To compute the velocity components $u_i$, 3 backward FFTs are required. Since the symmetric matrix $D_{ij} - S_{ij}$ is traceless, it has just 5 independent components.
• Hence, a total of only 8 FFTs are required per integration stage.
• Hence, a total of only 8 FFTs are required per integration stage.

• The effective pressure $p\delta_{ij} + S_{ij}$ is solved as usual from the inverse Laplacian of the force minus the nonlinearity.
The vorticity $\omega = \nabla \times u$ evolves according to

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla) \omega = (\omega \cdot \nabla) u + \nu \nabla^2 \omega + \nabla \times F,$$

where in 2D the vortex stretching term $(\omega \cdot \nabla) u$ vanishes and $\omega$ is normal to the plane of motion.
2D Basdevant Formulation: 4 FFTs

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• For $C^2$ velocity fields, the curl of the nonlinearity can be written in terms of $\tilde{D}_{ij} = D_{ij} - S_{ij}$:

$$\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_j} \tilde{D}_{2j} - \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_j} \tilde{D}_{1j} = \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) D_{12} + \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} (D_{22} - D_{11}),$$

on recalling that $S$ is diagonal and $S_{11} = S_{22}$.
2D Basdevant Formulation: 4 FFTs

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on recalling that $S$ is diagonal and $S_{11} = S_{22}$.

- The scalar vorticity $\omega$ thus evolves as

$$\frac{\partial \omega}{\partial t} + \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) (u_1 u_2) + \frac{\partial^2}{\partial x_1 \partial x_2} (u_2^2 - u_1^2) = \nu \nabla^2 \omega + \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}.$$
To compute $u_1$ and $u_2$ in physical space, we need 2 backward FFTs.
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• The quantities $u_1 u_2$ and $u_2^2 - u_1^2$ can then be calculated and then transformed to Fourier space with 2 additional forward FFTs.
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• The advective term in 2D can thus be calculated with just 4 FFTs.
3D Incompressible MHD: 17 FFTs

\[
\frac{\partial u_i}{\partial t} + \frac{\partial (D_{ij} - S_{ij})}{\partial x_j} = -\frac{\partial (p\delta_{ij} + S_{ij})}{\partial x_j} + \nu \frac{\partial^2 u_i}{\partial x_j^2},
\]

\[
\frac{\partial B_i}{\partial t} + \frac{\partial G_{ij}}{\partial x_j} = \eta \frac{\partial^2 B_i}{\partial x_j^2},
\]

where \( D_{ij} = u_i u_j - B_i B_j \), \( S_{ij} = \delta_{ij} \text{tr D}/3 \), and

\( G_{ij} = B_i u_j - u_i B_j \).

• The traceless matrix \( D_{ij} - S_{ij} \) has 8 independent components.
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- The traceless matrix \(D_{ij} - S_{ij}\) has 8 independent components.
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- The traceless matrix \( D_{ij} - S_{ij} \) has 8 independent components.
- The antisymmetric matrix \( G_{ij} \) has only 3.
- An additional 6 FFT calls are required to compute the components of \( \mathbf{u} \) and \( \mathbf{B} \) in \( x \) space.
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\frac{\partial u_i}{\partial t} + \frac{\partial (D_{ij} - S_{ij})}{\partial x_j} = -\frac{\partial (p\delta_{ij} + S_{ij})}{\partial x_j} + \nu \frac{\partial^2 u_i}{\partial x_j^2},
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- The antisymmetric matrix \( G_{ij} \) has only 3.
- An additional 6 FFT calls are required to compute the components of \( \mathbf{u} \) and \( \mathbf{B} \) in \( x \) space.
- The MHD nonlinearity can thus be computed with 17 FFT calls.
Discrete Cyclic Convolution

- The FFT provides an efficient tool for computing the \textit{discrete cyclic convolution}

\[
\sum_{p=0}^{N-1} F_p G_{k-p},
\]

where the vectors $F$ and $G$ have period $N$. 
Discrete Cyclic Convolution

• The FFT provides an efficient tool for computing the \emph{discrete cyclic convolution}:

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\sum_{p=0}^{N-1} F_p G_{k-p},
\]

where the vectors $F$ and $G$ have period $N$.

• The backward 1D \emph{discrete Fourier transform} of a complex vector \( \{F_k : k = 0, \ldots, N - 1\} \) is defined as

\[
f_j = \sum_{k=0}^{N-1} \zeta_N^{jk} F_k, \quad j = 0, \ldots, N - 1,
\]

where $\zeta_N = e^{2\pi i/N}$ denotes the \emph{Nth primitive root of unity}. 
Discrete Cyclic Convolution

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where \( \zeta_N = e^{2\pi i/N} \) denotes the \( N \)th primitive root of unity.

• The fast Fourier transform (FFT) method exploits the properties that \( \zeta_N^r = \zeta_{N/r} \) and \( \zeta_N^N = 1 \).
Convolution Theorem

\[
\sum_{j=0}^{N-1} f_j g_j \zeta_N^{-jk} = \sum_{j=0}^{N-1} \zeta_N^{-jk} \left( \sum_{p=0}^{N-1} \zeta_N^{jp} F_p \right) \left( \sum_{q=0}^{N-1} \zeta_N^{jq} G_q \right)
\]

\[
= \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} F_p G_q \sum_{j=0}^{N-1} \zeta_N^{-(k+p+q)j}
\]

\[
= N \sum_{s} \sum_{p=0}^{N-1} F_p G_{k-p+sN}.
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- The terms indexed by \( s \neq 0 \) are aliases; we need to remove them!
Convolution Theorem

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\]

\[
= \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} F_p G_q \sum_{j=0}^{N-1} \zeta_N^{(-k+p+q)j}
\]

\[
= N \sum_{s} \sum_{p=0}^{N-1} F_p G_{k-p+sN}.
\]

- The terms indexed by \( s \neq 0 \) are **aliases**; we need to remove them!

- If only the first \( m \) entries of the input vectors are nonzero, aliases can be avoided by **zero padding** input data vectors of length \( m \) to length \( N \geq 2m - 1 \).
Convolution Theorem

\[
\sum_{j=0}^{N-1} f_j g_j \zeta_N^{-jk} = \sum_{j=0}^{N-1} \zeta_N^{-jk} \left( \sum_{p=0}^{N-1} \zeta_N^{jp} F_p \right) \left( \sum_{q=0}^{N-1} \zeta_N^{jq} G_q \right)
\]

\[
= \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} F_p G_q \sum_{j=0}^{N-1} \zeta_N^{(-k+p+q)j}
\]

\[
= N \sum_{s} \sum_{p=0}^{N-1} F_p G_{k-p+sN}.
\]

• The terms indexed by \( s \neq 0 \) are **aliases**; we need to remove them!

• If only the first \( m \) entries of the input vectors are nonzero, aliases can be avoided by **zero padding** input data vectors of length \( m \) to length \( N \geq 2m - 1 \).

• **Explicit zero padding** prevents mode \( m - 1 \) from beating with itself, wrapping around to contaminate mode \( N = 0 \mod N \).
Implicit Dealiasing

Let $N = 2^m$. For $j = 0, \ldots, 2^m - 1$ we want to compute

$$f_j = \sum_{k=0}^{2^m-1} \zeta_{2^m}^{jk} F_k.$$
Implicit Dealiasing

• Let $N = 2m$. For $j = 0, \ldots, 2m - 1$ we want to compute

$$f_j = \sum_{k=0}^{2m-1} \zeta_{2m}^j F_k.$$

• If $F_k = 0$ for $k \geq m$, one can easily avoid looping over the unwanted zero Fourier modes by decimating in wavenumber:

$$f_{2\ell} = \sum_{k=0}^{m-1} \zeta_{2m}^{2\ell k} F_k = \sum_{k=0}^{m-1} \zeta_m^{\ell k} F_k,$$

$$f_{2\ell+1} = \sum_{k=0}^{m-1} \zeta_{2m}^{(2\ell+1)k} F_k = \sum_{k=0}^{m-1} \zeta_m^{\ell k} \zeta_{2m}^k F_k, \quad \ell = 0, 1, \ldots, m - 1.$$
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- This requires computing two subtransforms, each of size $m$, for an overall computational scaling of order $2m \log_2 m = N \log_2 m$. 
Parallelized multidimensional implicit dealiasing routines have been implemented as a software layer **FFTW++** (v 2.05) on top of the **FFTW** library under the Lesser GNU Public License:

http://fftwpp.sourceforge.net/

\[
\{ F_k \}_{k=0}^{m-1} \quad \{ G_k \}_{k=0}^{m-1}
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- Analytical bounds for random forcing provide a means to evaluate various heuristic turbulent subgrid (and supergrid!) models by characterizing the behaviour of the global attractor under these models.
References


