

# On the Global Attractor of 2D Incompressible Turbulence with Random Forcing

John C. Bowman and Pedram Emami  
Department of Mathematical and Statistical Sciences  
University of Alberta

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# Turbulence

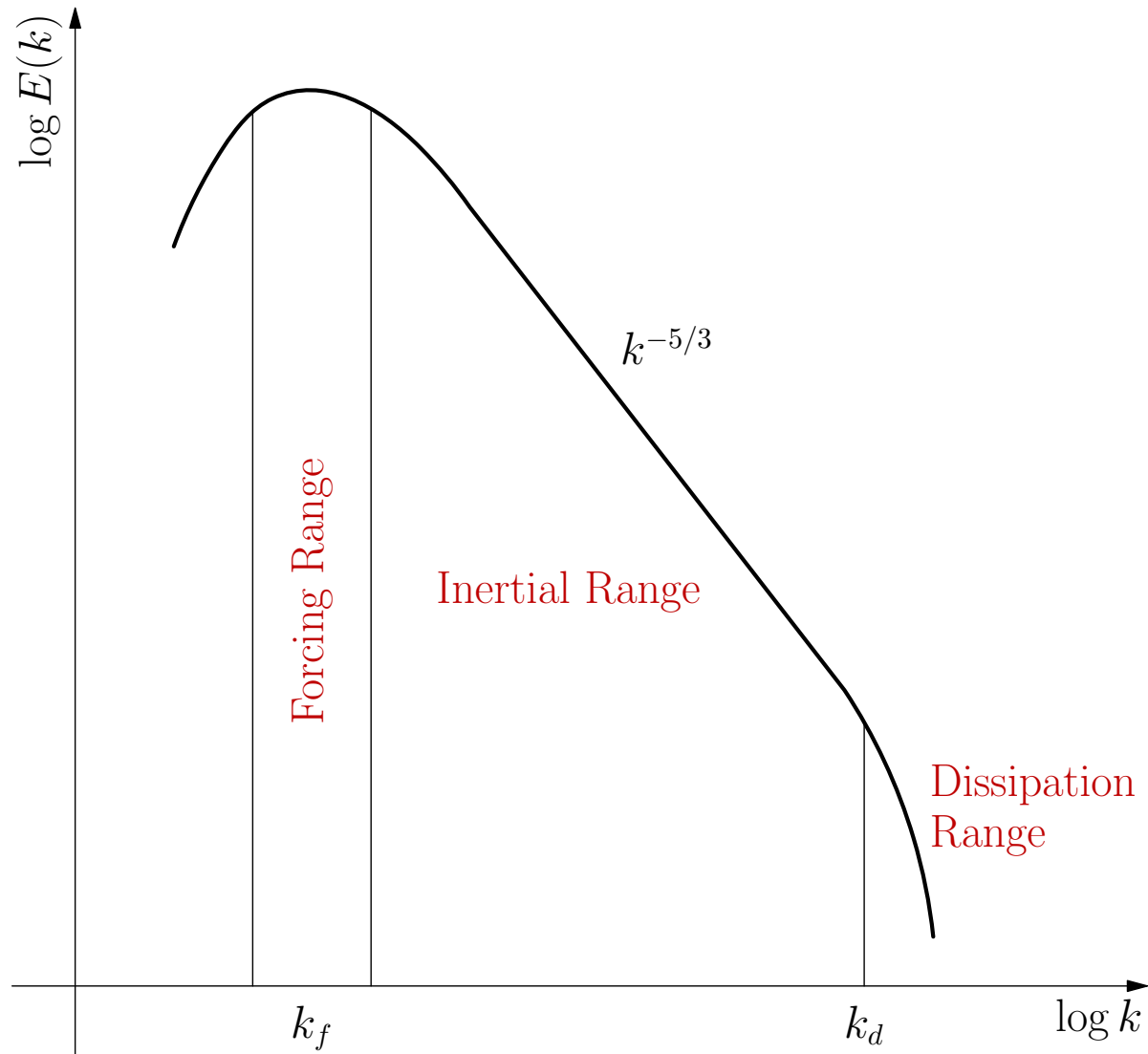
*Big whirls have little whirls that feed on their velocity, and little whirls have littler whirls and so on to viscosity...* [Richardson 1922]

- In 1941, Kolmogorov conjectured that the energy spectrum of 3D incompressible turbulence exhibits a self-similar power-law scaling characterized by a uniform *cascade* of energy to molecular (viscous) scales:

$$E(k) = C\epsilon^{2/3}k^{-5/3}.$$

- Here  $k$  is the Fourier wavenumber and  $E(k)$  is normalized so that  $\int E(k) dk$  is the total energy.
- Kolmogorov suggested that  $C$  might be a universal constant.

# 3D Energy Cascade



## 2D Incompressible Turbulence

- In 2D, where  $\mathbf{u}$  maps a plane normal to  $\hat{\mathbf{z}}$  to  $R^2$ , the **vorticity** vector  $\boldsymbol{\omega} = \nabla \times \mathbf{u}$  is always perpendicular to  $\mathbf{u}$ .
- Navier–Stokes equation for the scalar vorticity  $\omega = \hat{\mathbf{z}} \cdot \nabla \times \mathbf{u}$ :

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nu \nabla^2 \omega + f.$$

- The incompressibility condition  $\nabla \cdot \mathbf{u} = 0$  can be exploited to find  $\mathbf{u}$  in terms of  $\omega$ :

$$\nabla \omega \times \hat{\mathbf{z}} = \nabla \times \hat{\mathbf{z}} \omega = \nabla \times (\nabla \times \mathbf{u}) = \nabla (\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u} = -\nabla^2 \mathbf{u}.$$

- Thus  $\mathbf{u} = \hat{\mathbf{z}} \times \nabla \nabla^{-2} \omega$ . In Fourier space:

$$\frac{d\omega_{\mathbf{k}}}{dt} = S_{\mathbf{k}} - \nu k^2 \omega_{\mathbf{k}} + f_{\mathbf{k}},$$

$$\text{where } S_{\mathbf{k}} = \sum_{\mathbf{q}} \frac{\hat{\mathbf{z}} \times \mathbf{q} \cdot \mathbf{k}}{q^2} \overline{\omega_{\mathbf{q}}} \overline{\omega_{-\mathbf{k}-\mathbf{q}}} = \sum_{\mathbf{p}, \mathbf{q}} \frac{\epsilon_{\mathbf{k}\mathbf{p}\mathbf{q}}}{q^2} \overline{\omega_{\mathbf{p}}} \overline{\omega_{\mathbf{q}}}.$$

Here  $\epsilon_{kpq} \doteq \hat{\mathbf{z}} \cdot \mathbf{p} \times \mathbf{q} \delta_{\mathbf{k}+\mathbf{p}+\mathbf{q}}$  is antisymmetric under permutation of any two indices.

$$\frac{d\omega_{\mathbf{k}}}{dt} + \nu k^2 \omega_{\mathbf{k}} = \sum_{\mathbf{p}} \sum_{\mathbf{q}} \frac{\epsilon_{kpq}}{q^2} \overline{\omega_{\mathbf{p}}} \overline{\omega_{\mathbf{q}}} + f_{\mathbf{k}},$$

- When  $\nu = f_{\mathbf{k}} = 0$ :

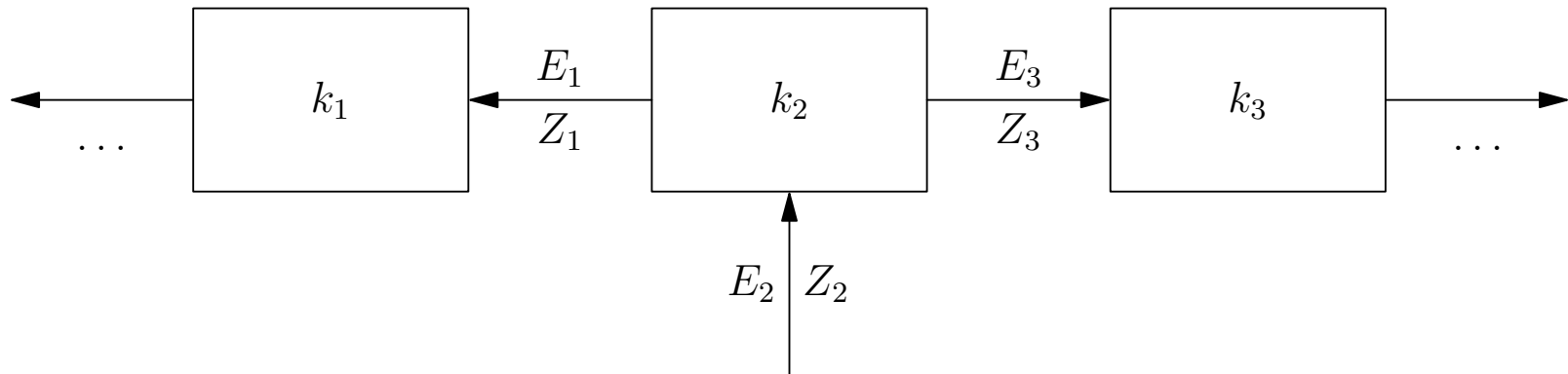
enstrophy  $Z = \frac{1}{2} \sum_{\mathbf{k}} |\omega_{\mathbf{k}}|^2$  and energy  $E = \frac{1}{2} \sum_{\mathbf{k}} \frac{|\omega_{\mathbf{k}}|^2}{k^2}$  are

conserved:

$$\frac{\epsilon_{kpq}}{q^2} \quad \text{antisymmetric in} \quad \mathbf{k} \leftrightarrow \mathbf{p},$$

$$\frac{1}{k^2} \frac{\epsilon_{kpq}}{q^2} \quad \text{antisymmetric in} \quad \mathbf{k} \leftrightarrow \mathbf{q}.$$

# Fjørtoft Dual Cascade Scenario



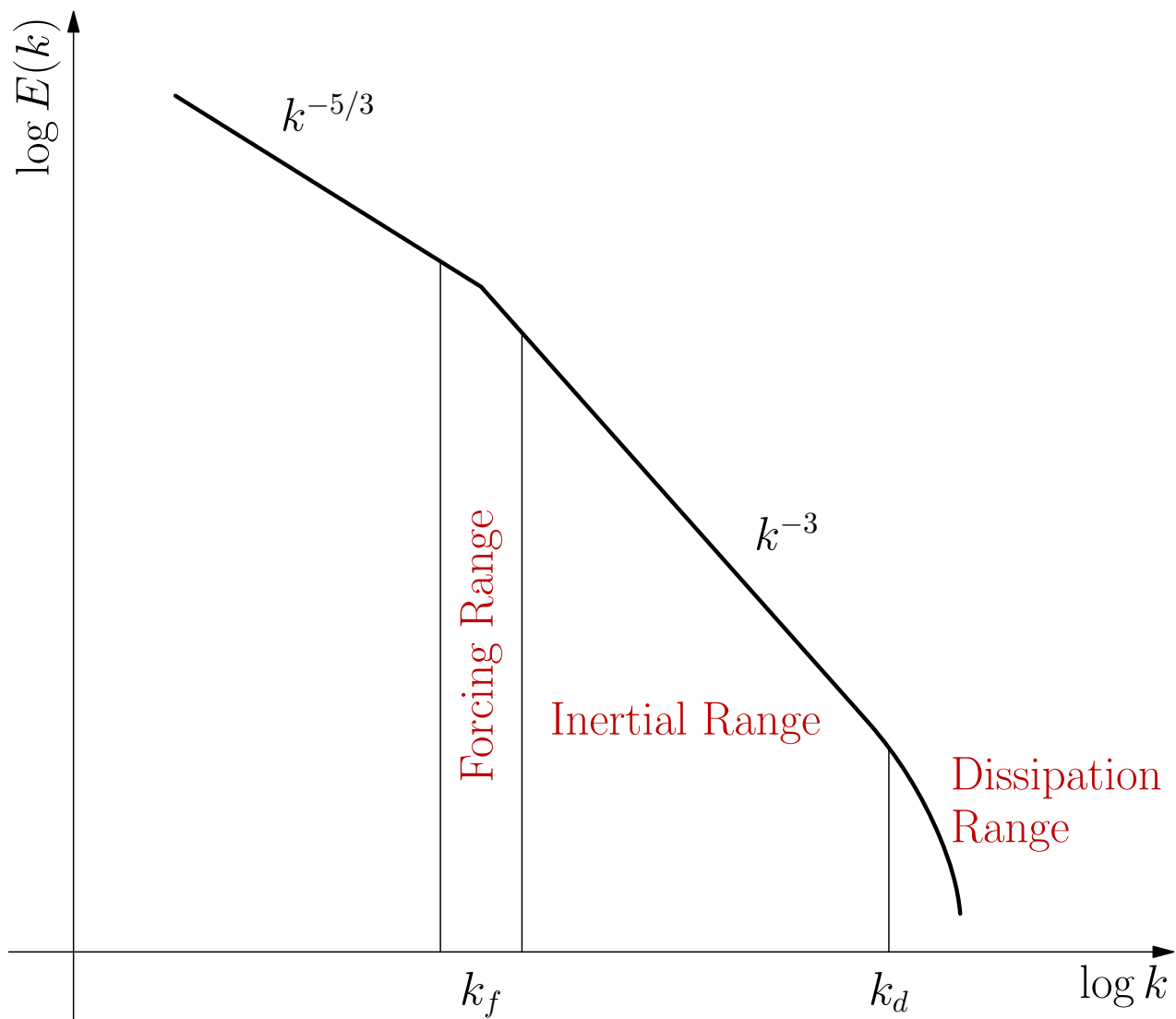
$$E_2 = E_1 + E_3, \quad Z_2 = Z_1 + Z_3, \quad Z_i \approx k_i^2 E_i.$$

- When  $k_1 = k$ ,  $k_2 = 2k$ , and  $k_3 = 4k$ :

$$E_1 \approx \frac{4}{5} E_2, \quad Z_1 \approx \frac{1}{5} Z_2, \quad E_3 \approx \frac{1}{5} E_2, \quad Z_3 \approx \frac{4}{5} Z_2.$$

- Fjørtoft [1953]: energy cascades to large scales and enstrophy cascades to small scales.

# 2D Energy Cascade



# 2D Turbulence: Mathematical Formulation

- Consider the Navier–Stokes equations for 2D incompressible homogeneous isotropic turbulence with density  $\rho = 1$ :

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} - \nu \nabla^2 \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P &= \mathbf{F}, \\ \nabla \cdot \mathbf{u} &= 0, \\ \int_{\Omega} \mathbf{u} \, d\mathbf{x} &= \mathbf{0}, \quad \int_{\Omega} \mathbf{F} \, d\mathbf{x} = \mathbf{0}, \\ \mathbf{u}(\mathbf{x}, 0) &= \mathbf{u}_0(\mathbf{x}),\end{aligned}$$

with  $\Omega = [0, 2\pi] \times [0, 2\pi]$  and periodic boundary conditions on  $\partial\Omega$ .

- Introduce the Hilbert space

$$H(\Omega) \doteq \text{cl} \left\{ \mathbf{u} \in (C^2(\Omega) \cap L^2(\Omega))^2 \mid \nabla \cdot \mathbf{u} = 0, \int_{\Omega} \mathbf{u} \, d\mathbf{x} = \mathbf{0} \right\}.$$

with inner product  $(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) \, d\mathbf{x}$  and  $L^2$  norm  $|\mathbf{u}| = (\mathbf{u}, \mathbf{u})^{1/2}$ .



- For  $\mathbf{u} \in H(\Omega)$ , the Navier–Stokes equations can be expressed:

$$\frac{d\mathbf{u}}{dt} - \nu \nabla^2 \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \mathbf{F}.$$

- Introduce  $A \doteq -\mathcal{P}(\nabla^2)$ ,  $\mathbf{f} \doteq \mathcal{P}(\mathbf{F})$ , and the **bilinear map**

$$\mathcal{B}(\mathbf{u}, \mathbf{u}) \doteq \mathcal{P}(\mathbf{u} \cdot \nabla \mathbf{u} + \nabla P),$$

where  $\mathcal{P}$  is the **Helmholtz–Leray projection operator** from  $(L^2(\Omega))^2$  to  $H(\Omega)$ :

$$\mathcal{P}(\mathbf{v}) \doteq \mathbf{v} - \nabla \nabla^{-2} \nabla \cdot \mathbf{v}, \quad \forall \mathbf{v} \in (L^2(\Omega))^2.$$

- The dynamical system can then be compactly written:

$$\frac{d\mathbf{u}}{dt} + \nu A\mathbf{u} + \mathcal{B}(\mathbf{u}, \mathbf{u}) = \mathbf{f}.$$

# Stokes Operator $A$

- The operator  $A = \mathcal{P}(-\nabla^2)$  is **positive semi-definite** and **self-adjoint**, with a compact inverse.
- On the periodic domain  $\Omega = [0, 2\pi] \times [0, 2\pi]$ , the eigenvalues of  $A$  are

$$\lambda = \mathbf{k} \cdot \mathbf{k}, \quad \mathbf{k} \in \mathbb{Z} \times \mathbb{Z} \setminus \{\mathbf{0}\}.$$

- The eigenvalues of  $A$  can be arranged as

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \quad \lambda_0 = 1$$

and its eigenvectors  $\mathbf{w}_i$ ,  $i \in \mathbb{N}_0$ , form an orthonormal basis for the Hilbert space  $H$ , upon which we can define any quotient power of  $A$ :

$$A^\alpha \mathbf{w}_j = \lambda_j^\alpha \mathbf{w}_j, \quad \alpha \in \mathbb{R}, \quad j \in \mathbb{N}_0.$$

# Subspace of Finite Enstrophy

- We define the subspace of  $H$  consisting of solutions with finite enstrophy:

$$V \doteq \left\{ \mathbf{u} \in H \mid \sum_{j=0}^{\infty} \lambda_j (\mathbf{u}, \mathbf{w}_j)^2 < \infty \right\}.$$

- Another suitable norm for elements  $\mathbf{u} \in V$  is

$$\|\mathbf{u}\| = \left| A^{1/2} \mathbf{u} \right| = \left( \int_{\Omega} \sum_{i=1}^2 \frac{\partial \mathbf{u}}{\partial x_i} \cdot \frac{\partial \mathbf{u}}{\partial x_i} \right)^{1/2} = \left( \sum_{j=0}^{\infty} \lambda_j (\mathbf{u}, \mathbf{w}_j)^2 \right)^{1/2}.$$

# Properties of the Bilinear Map

- We make use of the **antisymmetry**

$$(\mathcal{B}(\mathbf{u}, \mathbf{v}), \mathbf{w}) = -(\mathcal{B}(\mathbf{u}, \mathbf{w}), \mathbf{v}).$$

- In 2D, we also have **orthogonality**:

$$(\mathcal{B}(\mathbf{u}, \mathbf{u}), A\mathbf{u}) = 0$$

and the strong form of **enstrophy invariance**:

$$(\mathcal{B}(A\mathbf{v}, \mathbf{v}), \mathbf{u}) = (\mathcal{B}(\mathbf{u}, \mathbf{v}), A\mathbf{v}).$$

- In 2D the above properties imply the symmetry

$$(\mathcal{B}(A\mathbf{u}, \mathbf{u}), \mathbf{u}) + (\mathcal{B}(\mathbf{v}, A\mathbf{v}), \mathbf{u}) + (\mathcal{B}(\mathbf{v}, \mathbf{v}), A\mathbf{v}) = 0.$$

# Dynamical Behaviour

- Our starting point is the incompressible 2D Navier–Stokes equation with periodic boundary conditions:

$$\frac{d\mathbf{u}}{dt} + \nu A\mathbf{u} + \mathcal{B}(\mathbf{u}, \mathbf{u}) = \mathbf{f}, \quad \mathbf{u} \in H.$$

- Take the inner product with  $\mathbf{u}$  (respectively  $A\mathbf{u}$ ):

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\mathbf{u}(t)|^2 + \nu \|\mathbf{u}(t)\|^2 &= (\mathbf{f}, \mathbf{u}(t)), \\ \frac{1}{2} \frac{d}{dt} \|\mathbf{u}(t)\|^2 + \nu |A\mathbf{u}(t)|^2 &= (\mathbf{f}, A\mathbf{u}(t)). \end{aligned}$$

- The Cauchy–Schwarz and Poincaré inequalities yield

$$(\mathbf{f}, \mathbf{u}(t)) \leq |\mathbf{f}| |\mathbf{u}(t)| \quad \text{and} \quad |\mathbf{u}(t)| \leq \|\mathbf{u}(t)\|.$$

- Since the existence and uniqueness for solutions to the 2D Navier–Stokes equation has been proven, a global attractor can be defined [Ladyzhenskaya 1975], [Foias & Temam 1979].

# Dynamical Behaviour: Constant Forcing

- If the force  $\mathbf{f}$  is constant with respect to time, a **Gronwall inequality** can be exploited:

$$|\mathbf{u}(t)|^2 \leq e^{-\nu t} |\mathbf{u}(0)|^2 + (1 - e^{-\nu t}) \left( \frac{|\mathbf{f}|}{\nu} \right)^2.$$

- Defining a nondimensional **Grashof number**  $G = \frac{|\mathbf{f}|}{\nu^2}$ , the above inequality can be simplified to

$$|\mathbf{u}(t)|^2 \leq e^{-\nu t} |\mathbf{u}(0)|^2 + (1 - e^{-\nu t}) \nu^2 G^2.$$

- Similarly,

$$\|\mathbf{u}(t)\|^2 \leq e^{-\nu t} \|\mathbf{u}(0)\|^2 + (1 - e^{-\nu t}) \nu^2 G^2.$$

- Being on the attractor thus requires

$$|\mathbf{u}| \leq \nu G \quad \text{and} \quad \|\mathbf{u}\| \leq \nu G.$$

# Attractor Set $\mathcal{A}$

- Let  $S$  be the solution operator:

$$S(t)\mathbf{u}_0 = \mathbf{u}(t), \quad \mathbf{u}_0 = \mathbf{u}(0),$$

where  $\mathbf{u}(t)$  is the unique solution of the Navier–Stokes equations.

- The closed ball  $\mathfrak{B}$  of radius  $\nu G$  about the origin in the space  $V$  is a bounded absorbing set in  $H$ .
- That is, for any bounded set  $\mathfrak{B}'$  there exists a time  $t_0$  such that

$$S(t)\mathfrak{B}' \subset \mathfrak{B}, \quad \forall t \geq t_0.$$

- We can then construct the global attractor

$$\mathcal{A} = \bigcap_{t \geq 0} S(t)\mathfrak{B},$$

so  $\mathcal{A}$  is the largest bounded, invariant set such that  $S(t)\mathcal{A} = \mathcal{A}$  for all  $t \geq 0$ .

## $Z$ - $E$ Plane Bounds: Constant Forcing

- A trivial lower bound is provided by the Poincaré inequality:

$$|u|^2 \leq \|u\|^2 \quad \Rightarrow \quad E \leq Z.$$

- An upper bound is given by

**Theorem 1 (Dascalu, Foias, and Jolly [2005])**

*For all  $u \in \mathcal{A}$ ,*

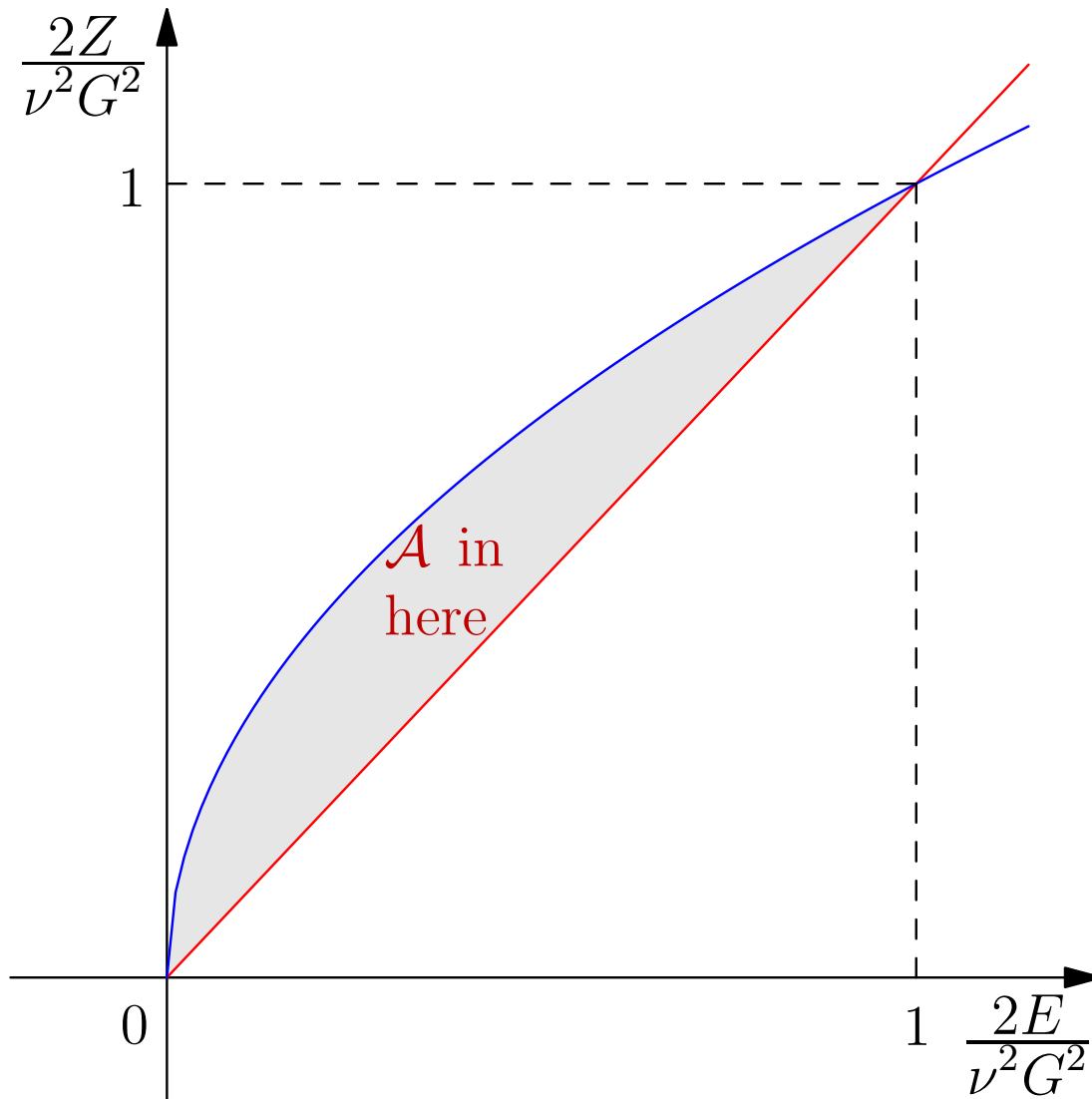
$$\|u\|^2 \leq \frac{|f|}{\nu} |u|.$$

- That is,

$$Z \leq \nu G \sqrt{E}.$$



# $Z$ - $E$ Plane Bounds: Constant Forcing



# Extended Norm: Random Forcing

- For a random variable  $\alpha$ , with probability density function  $P$ , define the ensemble average

$$\langle \alpha \rangle = \int_{-\infty}^{\infty} \alpha \left( \frac{dP}{d\zeta} \right) d\zeta.$$

- The extended inner product is

$$(\mathbf{u}, \mathbf{v})_{\tilde{\omega}} \doteq \int_{\Omega} \langle \mathbf{u} \cdot \mathbf{v} \rangle d\mathbf{x} = \int_{\Omega} \left( \int_{-\infty}^{\infty} \mathbf{u} \cdot \mathbf{v} \frac{dP}{d\zeta} d\zeta \right) d\mathbf{x},$$

with norm

$$|\mathbf{f}|_{\tilde{\omega}} \doteq \left( \int_{\Omega} \langle |\mathbf{f}|^2 \rangle d\mathbf{x} \right)^{1/2}.$$

# Dynamical Behaviour: Random Forcing

- Energy balance:

$$\frac{1}{2} \frac{d}{dt} |\mathbf{u}|^2 + \nu (A\mathbf{u}, \mathbf{u}) + (\mathcal{B}(\mathbf{u}, \mathbf{u}), \mathbf{u}) = (\mathbf{f}, \mathbf{u}) \doteq \epsilon,$$

where  $\epsilon$  is the rate of energy injection.

- From the energy conservation identity  $(\mathcal{B}(\mathbf{u}, \mathbf{u}), \mathbf{u}) = 0$ ,

$$\frac{1}{2} \frac{d}{dt} |\mathbf{u}|^2 + \nu \|\mathbf{u}\|^2 = \epsilon.$$

- The Poincaré inequality  $\|\mathbf{u}\| \geq |\mathbf{u}|$  leads to

$$\frac{1}{2} \frac{d}{dt} |\mathbf{u}|^2 \leq \epsilon - \nu |\mathbf{u}|^2,$$

which implies that  $|\mathbf{u}(t)|^2 \leq e^{-2\nu t} |\mathbf{u}(0)|^2 + \left( \frac{1 - e^{-2\nu t}}{\nu} \right) \epsilon$ .

- So for every  $\mathbf{u} \in \mathcal{A}$ , we expect  $|\mathbf{u}(t)|^2 \leq \epsilon/\nu$ .

- From  $|\mathbf{u}(t)| \leq \sqrt{\epsilon/\nu}$  we then obtain a lower bound for  $|\mathbf{f}|$ :

$$\sqrt{\nu\epsilon} \leq \frac{\epsilon}{|\mathbf{u}|} = \frac{(\mathbf{f}, \mathbf{u})}{|\mathbf{u}|} \leq \frac{|\mathbf{f}||\mathbf{u}|}{|\mathbf{u}|} = |\mathbf{f}|.$$

- It is convenient to use this lower bound for  $|\mathbf{f}|$  to define a lower bound for the Grashof number  $G = |\mathbf{f}|/\nu^2$ , which we use as the normalization  $\tilde{G}$  for random forcing:

$$\tilde{G} = \sqrt{\frac{\epsilon}{\nu^3}}.$$

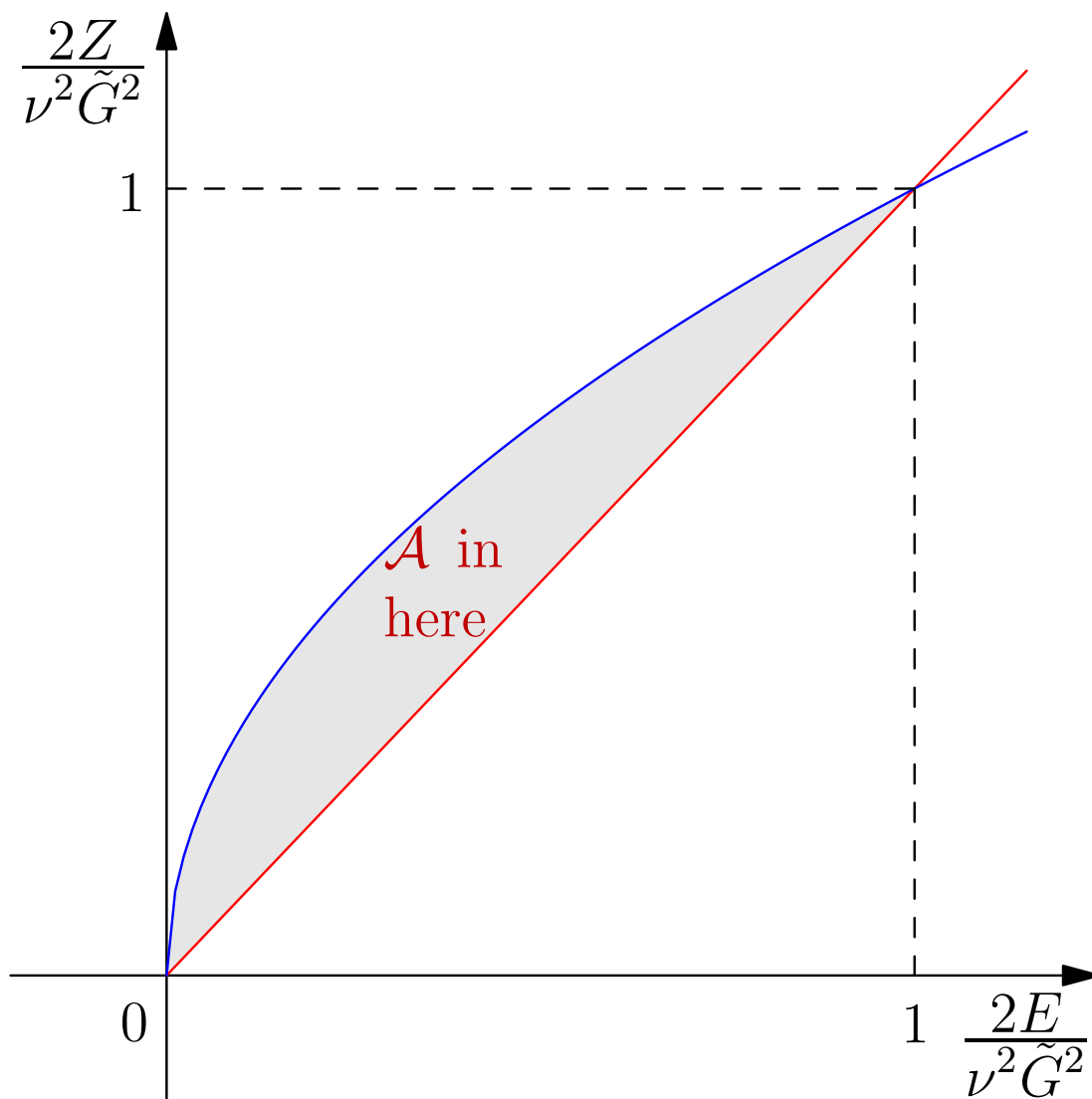
- We recently proved the following theorem (JDE 2018):

**Theorem 2 (Emami & Bowman [2018])** *For all  $\mathbf{u} \in \mathcal{A}$  with energy injection rate  $\epsilon$ ,*

$$\|\mathbf{u}\|^2 \leq \sqrt{\frac{\epsilon}{\nu}} |\mathbf{u}|.$$

- This leads to the **same form** as for a constant force:  $Z \leq \nu\tilde{G}\sqrt{E}$ .

# $Z$ - $E$ Plane Bounds: Random Forcing



# DNS code

- We have released a highly optimized 2D pseudospectral code in C++: <https://github.com/dealias/dns>.
- It uses our **FFTW++** library to implicitly dealias the advective convolution, while exploiting Hermitian symmetry [Bowman & Roberts 2011], [Roberts & Bowman 2018].
- Advanced computer memory management, such as implicit padding, memory alignment, and dynamic moment averaging allow **DNS** to attain its extreme performance.
- The formulation proposed by **Basdevant [1983]** is used to reduce the number of FFTs required for 2D (3D) incompressible turbulence to 4 (8).
- We also include simplified 2D (146 lines) and 3D (287 lines) versions called **ProtoDNS** for educational purposes: <https://github.com/dealias/dns/tree/master/protodns>.

# Dynamic Moment Averaging

- Advantageous to precompute time-integrated moments like

$$M_n(t) = \int_0^t |\omega_{\mathbf{k}}(\tau)|^n d\tau.$$

- This can be accomplished done by evolving

$$\frac{dM_n}{dt} = |\omega_{\mathbf{k}}|^n,$$

along with the vorticity  $\omega_{\mathbf{k}}$  itself, using *the same* temporal discretization.

- These evolved quantities  $M_n$  can be used to extract accurate statistical averages during the post-processing phase, once the saturation time  $t_1$  has been determined by the user:

$$\int_{t_1}^{t_2} |\omega_{\mathbf{k}}|^n(\tau) d\tau = M_n(t_2) - M_n(t_1).$$

# Enstrophy Balance

$$\frac{\partial \omega_{\mathbf{k}}}{\partial t} + \nu k^2 \omega_{\mathbf{k}} = S_{\mathbf{k}} + f_{\mathbf{k}},$$

- Multiply by  $\omega_{\mathbf{k}}^*$  and integrate over wavenumber angle  $\Rightarrow$  enstrophy spectrum  $Z(k)$  evolves as:

$$\frac{\partial}{\partial t} Z(k) + 2\nu k^2 Z(k) = 2T(k) + G(k),$$

where  $T(k)$  and  $G(k)$  are the corresponding angular averages of  $\text{Re} \langle S_{\mathbf{k}} \omega_{\mathbf{k}}^* \rangle$  and  $\text{Re} \langle f_{\mathbf{k}} \omega_{\mathbf{k}}^* \rangle$ .



# Nonlinear Enstrophy Transfer Function

$$\frac{\partial}{\partial t} Z(k) + 2\nu k^2 Z(k) = 2T(k) + G(k).$$

- Let

$$\Pi(k) \doteq 2 \int_k^\infty T(p) dp$$

represent the nonlinear transfer of enstrophy into  $[k, \infty)$ .

- Integrate from  $k$  to  $\infty$ :

$$\frac{d}{dt} \int_k^\infty Z(p) dp = \Pi(k) - \epsilon_Z(k),$$

where  $\epsilon_Z(k) \doteq 2\nu \int_k^\infty p^2 Z(p) dp - \int_k^\infty G(p) dp$  is the total enstrophy transfer, via dissipation and forcing, **out** of wavenumbers higher than  $k$ .

- A positive (negative) value for  $\Pi(k)$  represents a flow of enstrophy to wavenumbers higher (lower) than  $k$ .
- When  $\nu = 0$  and  $f_{\mathbf{k}} = 0$ :

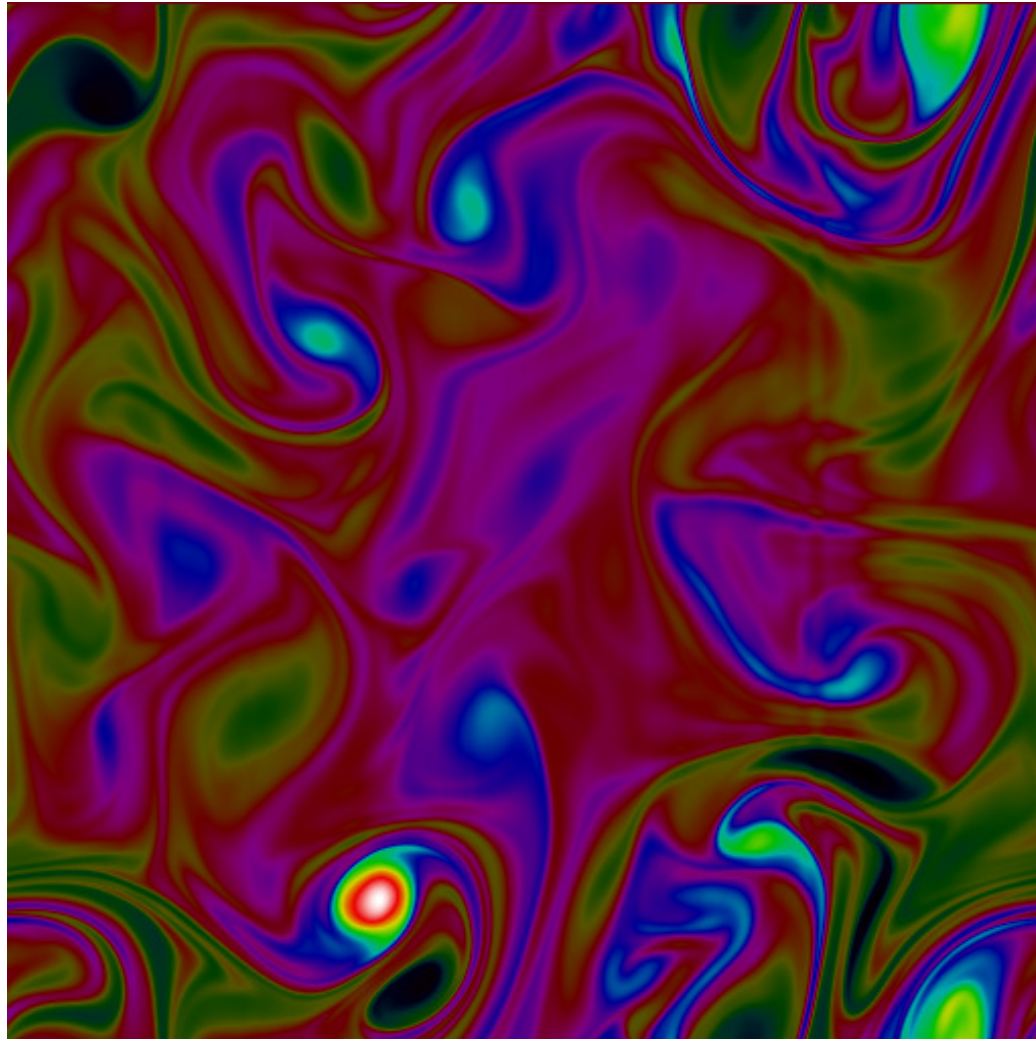
$$0 = \frac{d}{dt} \int_0^\infty Z(p) dp = 2 \int_0^\infty T(p) dp,$$

so that

$$\Pi(k) = 2 \int_k^\infty T(p) dp = -2 \int_0^k T(p) dp.$$

- Note that  $\Pi(0) = \Pi(\infty) = 0$ .
- In a steady state,  $\Pi(k) = \epsilon_Z(k)$ .
- This provides an excellent numerical diagnostic for determining the saturation time  $t_1$ .

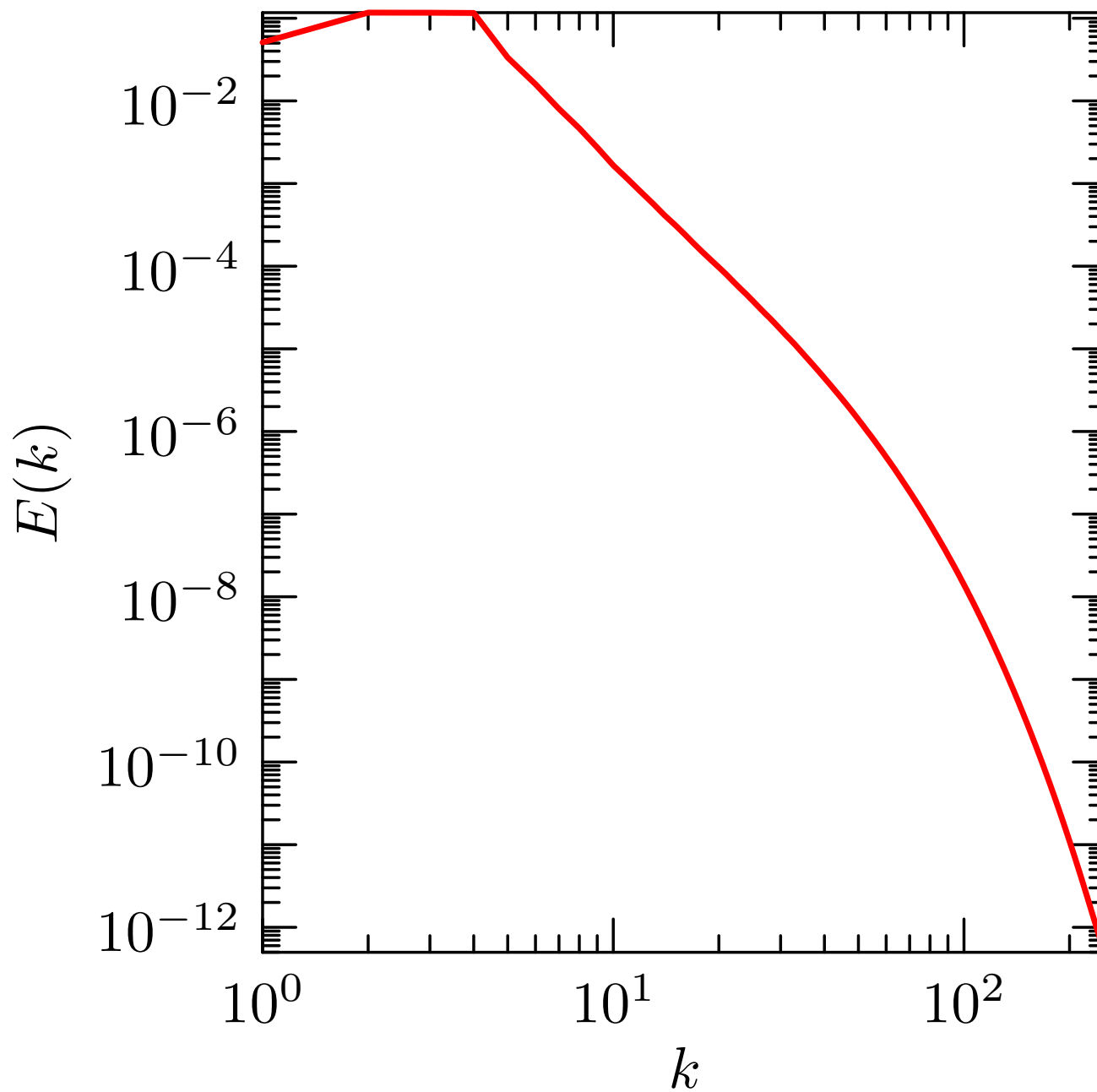
# Vorticity Field with Hypoviscosity



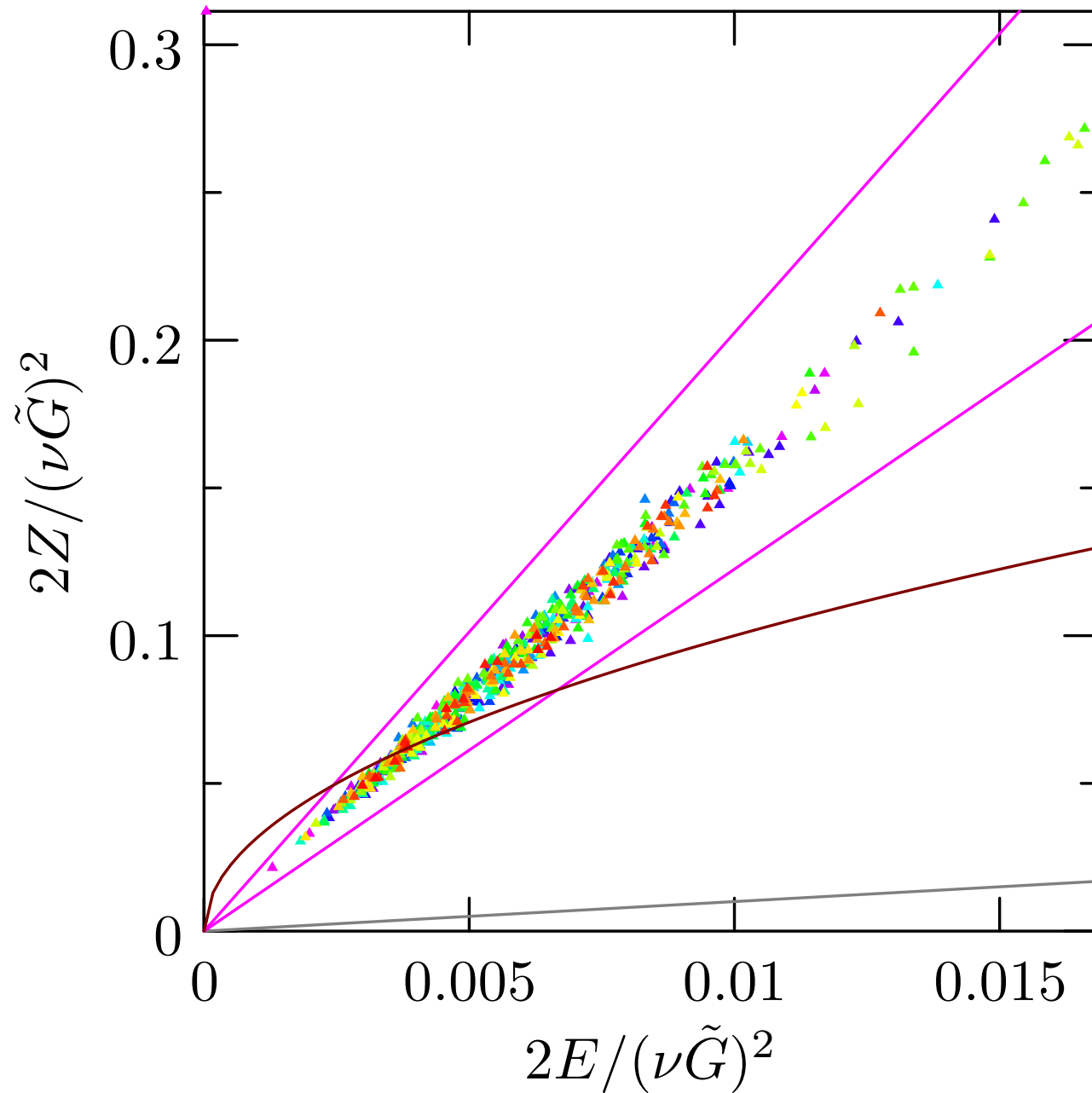
-10      0      10      20

$\omega$

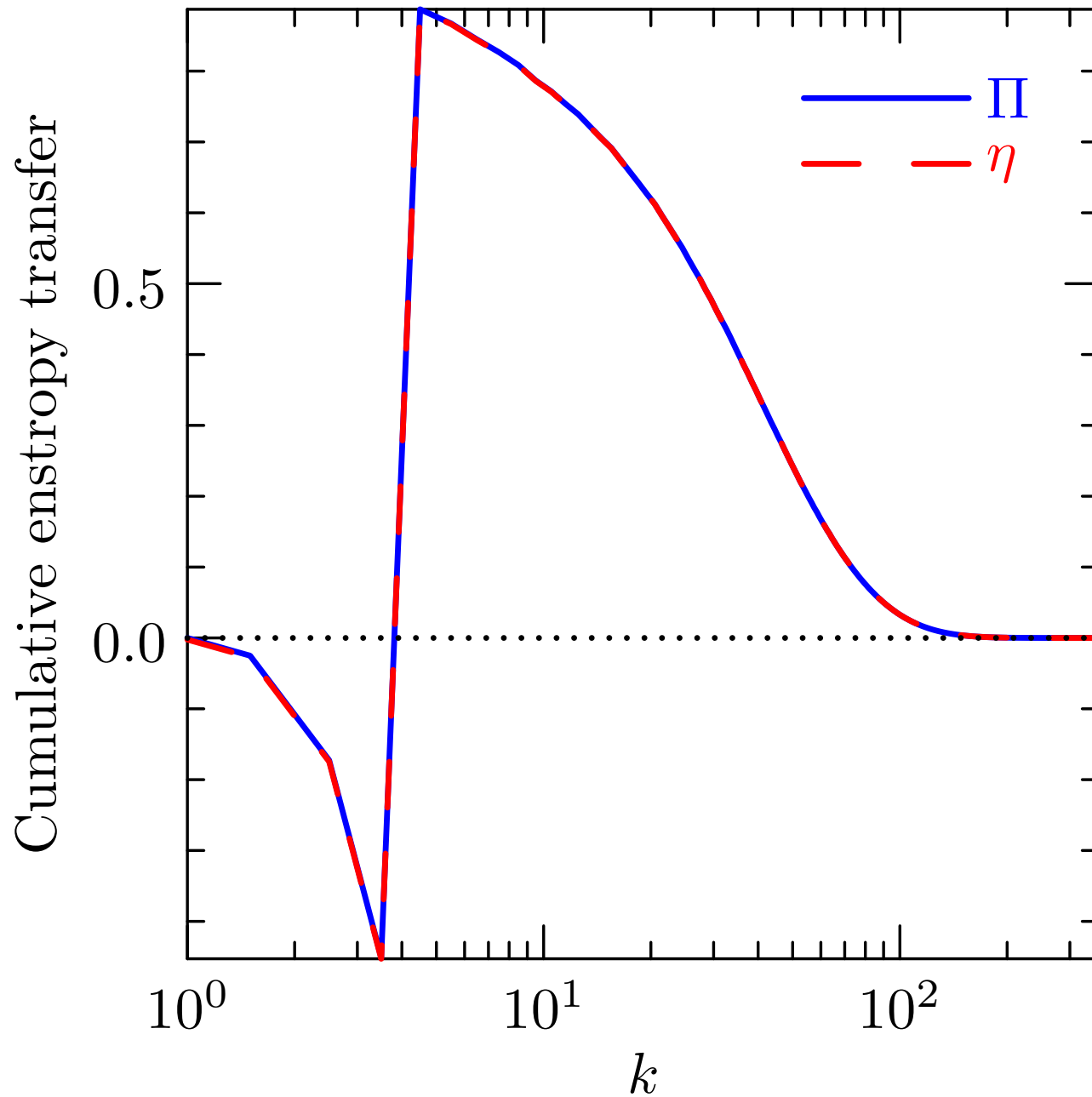
# Energy Spectrum with Hypoviscosity



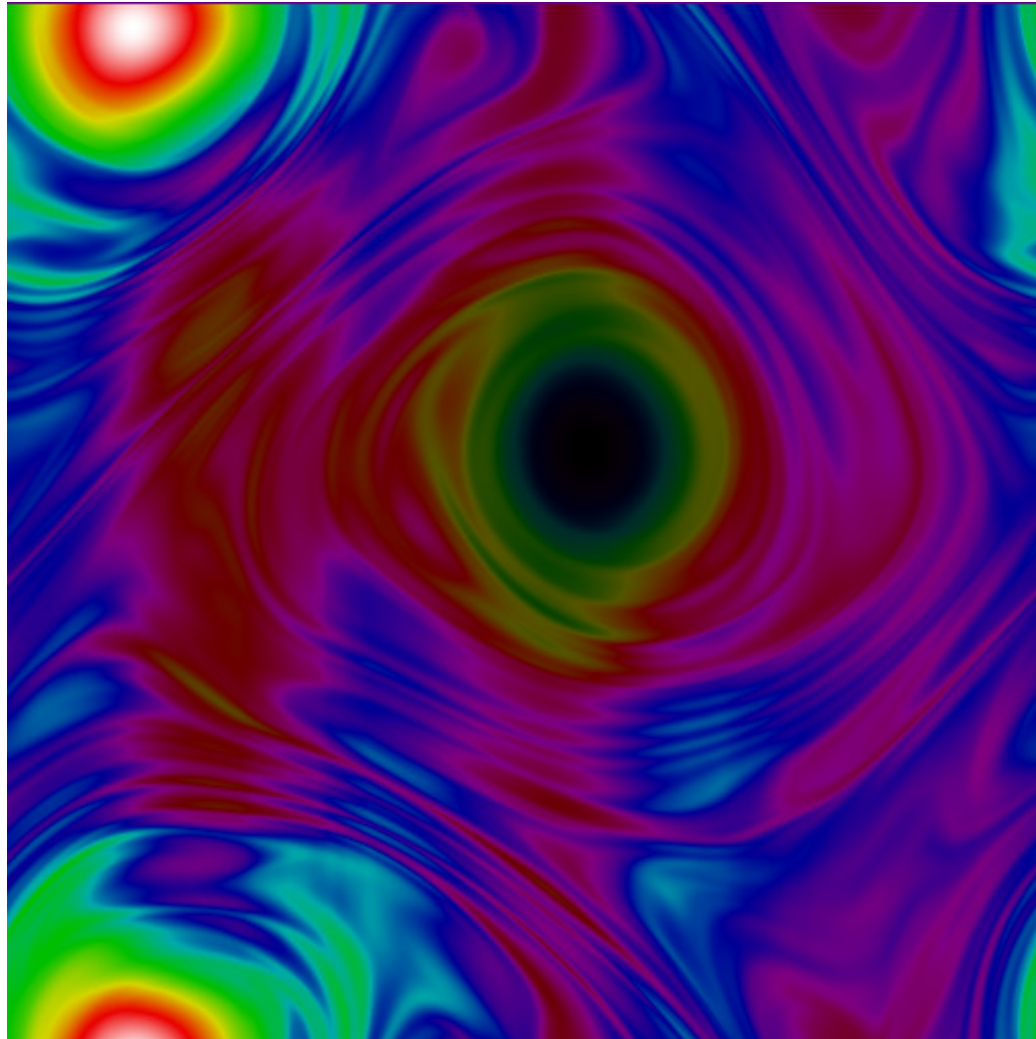
Bounds in the  $Z-E$  plane for random forcing.



# Energy Transfer with Hypoviscosity

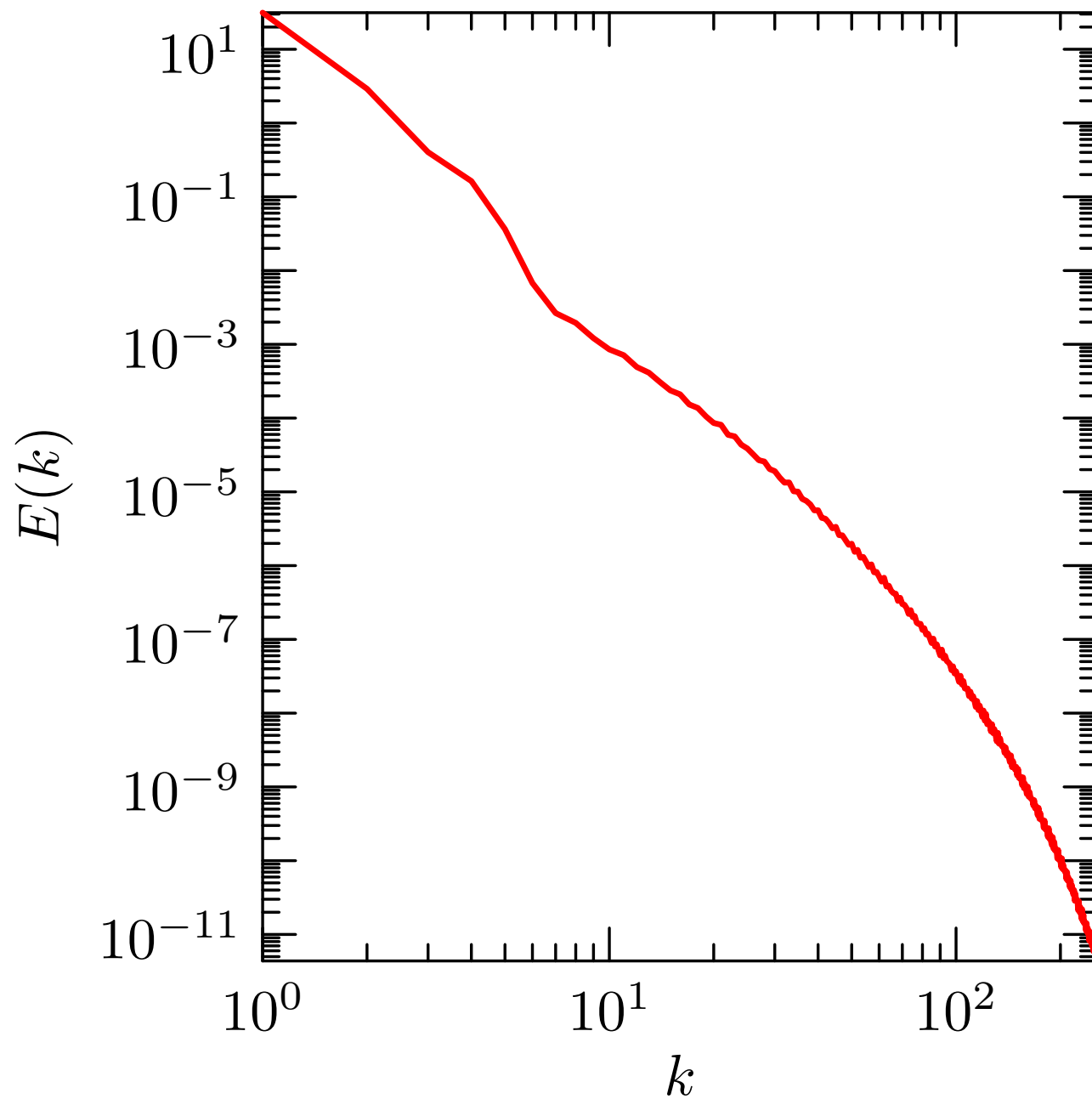


# Vorticity Field without Hypoviscosity



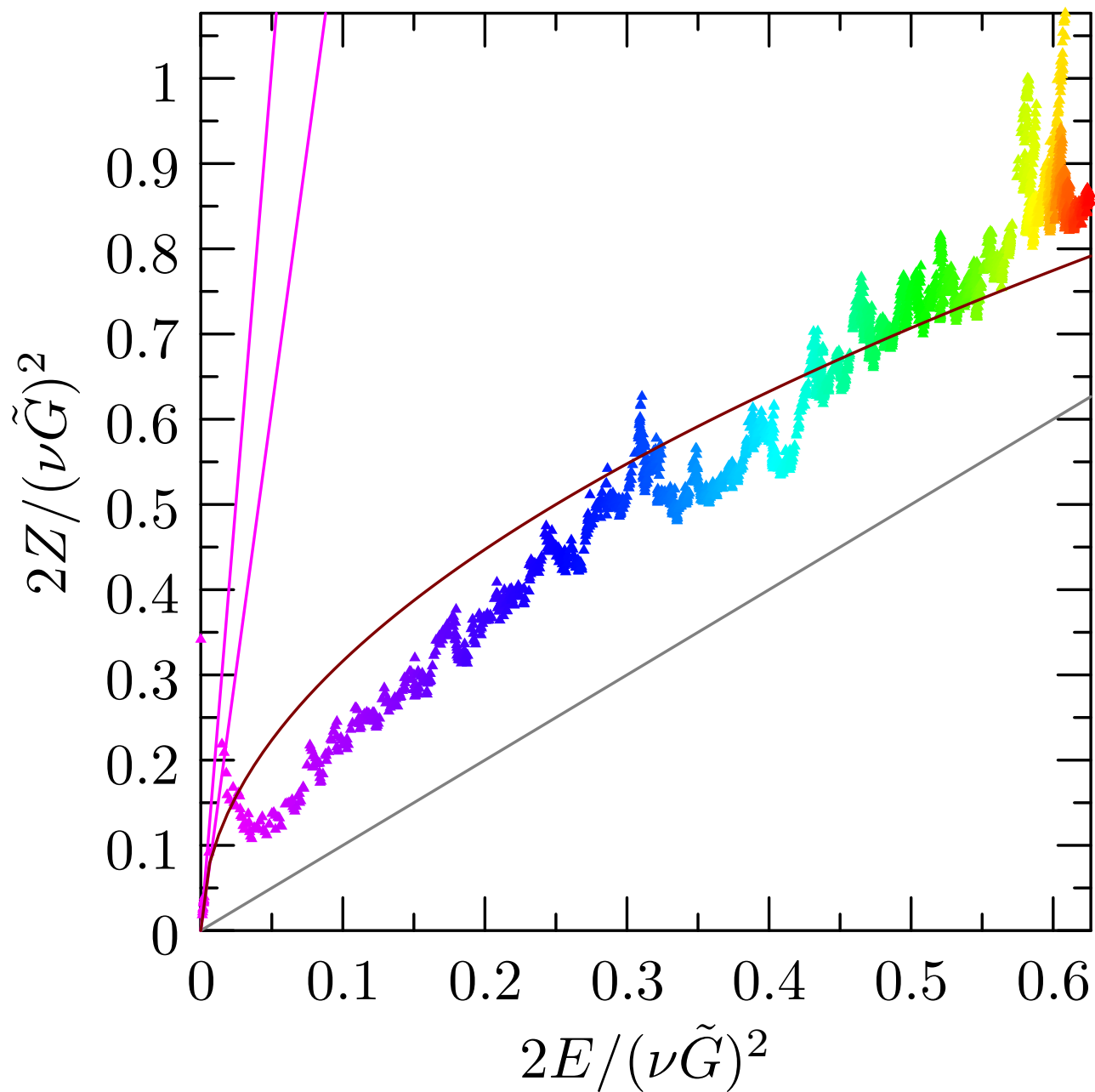
-25      0      25  
 $\omega$

# Energy Spectrum without Hypoviscosity

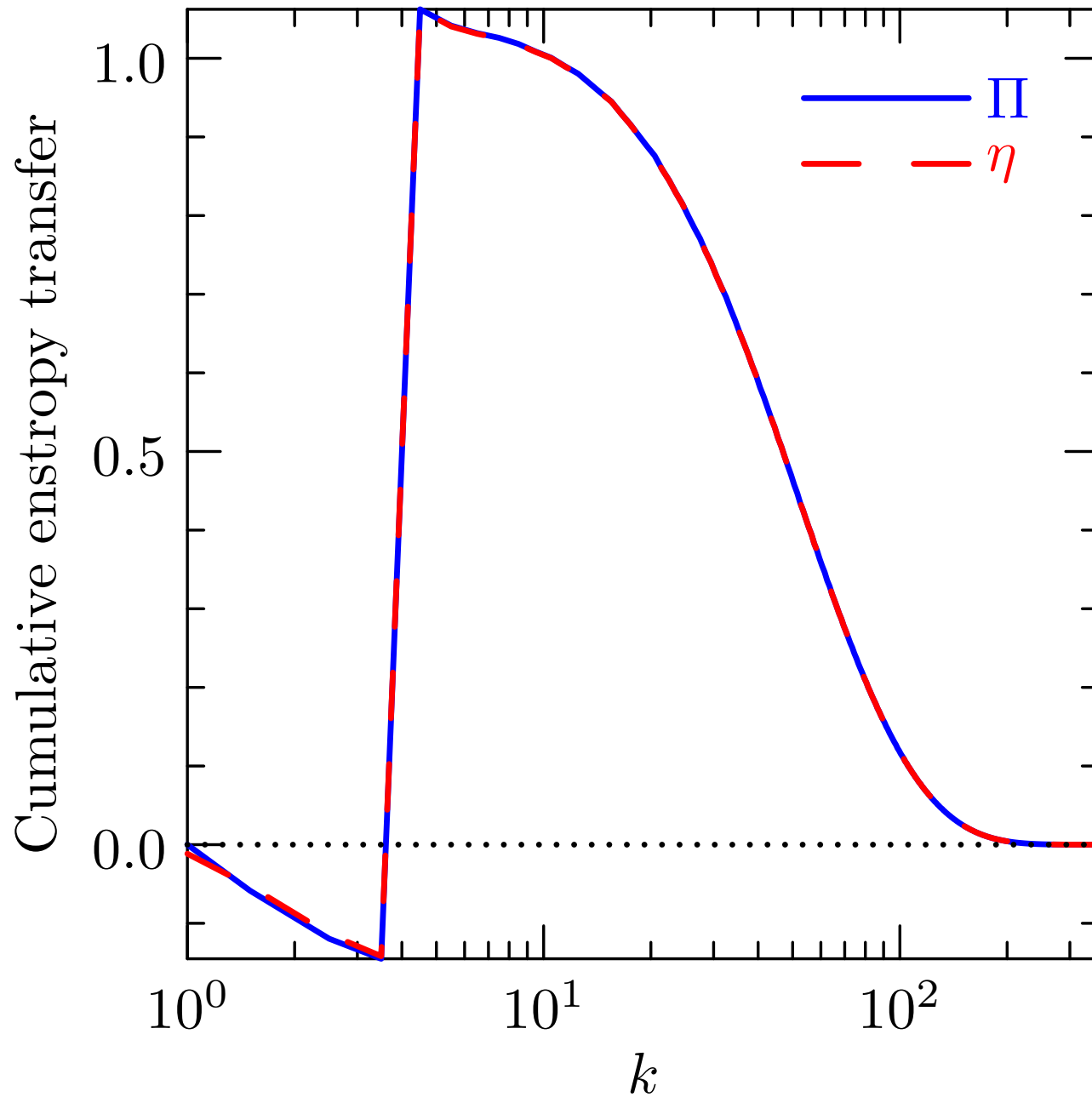




Bounds in the  $Z-E$  plane for random forcing.



# Energy Transfer without Hypoviscosity



## Special Case: White-Noise Forcing

- The Fourier transform of an isotropic Gaussian white-noise solenoidal force  $\mathbf{f}$  has the form

$$\mathbf{f}_{\mathbf{k}}(t) = F_{\mathbf{k}} \left( \mathbf{1} - \frac{\mathbf{k}\mathbf{k}}{k^2} \right) \cdot \boldsymbol{\xi}_{\mathbf{k}}(t), \quad \mathbf{k} \cdot \mathbf{f}_{\mathbf{k}} = 0,$$

where  $F_{\mathbf{k}}$  is a real number and  $\boldsymbol{\xi}_{\mathbf{k}}(t)$  is a unit central real Gaussian random 2D vector that satisfies

$$\langle \boldsymbol{\xi}_{\mathbf{k}}(t) \boldsymbol{\xi}_{\mathbf{k}'}(t') \rangle = \delta_{\mathbf{k}\mathbf{k}'} \mathbf{1} \delta(t - t').$$

- This implies

$$\langle \mathbf{f}_{\mathbf{k}}(t) \cdot \mathbf{f}_{\mathbf{k}'}(t') \rangle = F_{\mathbf{k}}^2 \delta_{\mathbf{k},\mathbf{k}'} \delta(t - t').$$

## Special Case: White-Noise Forcing

- To prescribe the forcing amplitude  $F_{\mathbf{k}}$  in terms of  $\epsilon$ :

**Theorem 3 (Novikov [1964])** *If  $f(\mathbf{x}, t)$  is a Gaussian process, and  $u$  is a functional of  $f$ , then*

$$\langle f(\mathbf{x}, t)u(f) \rangle = \int \int \langle f(\mathbf{x}, t)f(\mathbf{x}', t') \rangle \left\langle \frac{\delta u(\mathbf{x}, t)}{\delta f(\mathbf{x}', t')} \right\rangle d\mathbf{x}' dt'.$$

- For white-noise forcing:

$$\begin{aligned} \epsilon &= \text{Re} \sum_{\mathbf{k}} \langle \mathbf{f}_{\mathbf{k}}(t) \cdot \bar{\mathbf{u}}_{\mathbf{k}}(t) \rangle = \text{Re} \sum_{\mathbf{k}, \mathbf{k}'} \int \langle \mathbf{f}_{\mathbf{k}}(t) \bar{\mathbf{f}}_{\mathbf{k}'}(t') \rangle : \left\langle \frac{\delta \bar{\mathbf{u}}_{\mathbf{k}}(t)}{\delta \bar{\mathbf{f}}_{\mathbf{k}'}(t')} \right\rangle dt' \\ &= \sum_{\mathbf{k}} F_{\mathbf{k}}^2 \left( \mathbf{1} - \frac{\mathbf{k}\mathbf{k}}{k^2} \right) : \left( \mathbf{1} - \frac{\mathbf{k}\mathbf{k}}{k^2} \right) H(0) \\ &= \frac{1}{2} \sum_{\mathbf{k}} F_{\mathbf{k}}^2, \end{aligned}$$

on noting that  $H(0) = 1/2$ .

# White-Noise Forcing: Implementation

- At the end of each time-step, we implement the contribution of white noise forcing with the discretization

$$\omega_{\mathbf{k},n+1} = \omega_{\mathbf{k},n} + \sqrt{2\tau\eta_{\mathbf{k}}} \xi,$$

where  $\xi$  is a unit complex Gaussian random number with  $\langle \xi \rangle = 0$  and  $\langle |\xi|^2 \rangle = 1$ .

- This yields the mean enstrophy injection

$$\frac{\langle |\omega_{\mathbf{k},n+1}|^2 - |\omega_{\mathbf{k},n}|^2 \rangle}{2\tau} = \eta_{\mathbf{k}}.$$

## 3D Basdevant Formulation: 8 FFTs

- Using incompressibility, the 3D momentum equation can be written in terms of the symmetric tensor  $D_{ij} = u_i u_j$ :

$$\frac{\partial u_i}{\partial t} + \frac{\partial D_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + F_i.$$

- Naive implementation: 3 backward FFTs to compute the velocity components from their spectral representations, 6 forward FFTs of the independent components of  $D_{ij}$ .
- **Basdevant [1983]**: avoid one FFT by subtracting the divergence of the symmetric matrix  $S_{ij} = \delta_{ij} \text{tr } D/3$  from both sides:

$$\frac{\partial u_i}{\partial t} + \frac{\partial (D_{ij} - S_{ij})}{\partial x_j} = -\frac{\partial (p\delta_{ij} + S_{ij})}{\partial x_j} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + F_i.$$

- To compute the velocity components  $u_i$ , 3 backward FFTs are required. Since the symmetric matrix  $D_{ij} - S_{ij}$  is traceless, it has just 5 independent components.

- Hence, a total of only 8 FFTs are required per integration stage.
- The effective pressure  $p\delta_{ij} + S_{ij}$  is solved as usual from the inverse Laplacian of the force minus the nonlinearity.

## 2D Basdevant Formulation: 4 FFTs

- The vorticity  $\boldsymbol{\omega} = \nabla \times \mathbf{u}$  evolves according to

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla) \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega} + \nabla \times \mathbf{F},$$

where in 2D the vortex stretching term  $(\boldsymbol{\omega} \cdot \nabla) \mathbf{u}$  vanishes and  $\boldsymbol{\omega}$  is normal to the plane of motion.

- For  $C^2$  velocity fields, the curl of the nonlinearity can be written in terms of  $\tilde{D}_{ij} \doteq D_{ij} - S_{ij}$ :

$$\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_j} \tilde{D}_{2j} - \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_j} \tilde{D}_{1j} = \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) D_{12} + \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} (D_{22} - D_{11}),$$

on recalling that  $S$  is diagonal and  $S_{11} = S_{22}$ .

- The scalar vorticity  $\omega$  thus evolves as

$$\frac{\partial \omega}{\partial t} + \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) (u_1 u_2) + \frac{\partial^2}{\partial x_1 \partial x_2} (u_2^2 - u_1^2) = \nu \nabla^2 \omega + \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}.$$



- To compute  $u_1$  and  $u_2$  in physical space, we need 2 backward FFTs.
- The quantities  $u_1 u_2$  and  $u_2^2 - u_1^2$  can then be calculated and then transformed to Fourier space with 2 additional forward FFTs.
- The advective term in 2D can thus be calculated with just 4 FFTs.

## 3D Incompressible MHD: 17 FFTs

$$\frac{\partial u_i}{\partial t} + \frac{\partial(D_{ij} - S_{ij})}{\partial x_j} = -\frac{\partial(p\delta_{ij} + S_{ij})}{\partial x_j} + \nu \frac{\partial^2 u_i}{\partial x_j^2},$$
$$\frac{\partial B_i}{\partial t} + \frac{\partial G_{ij}}{\partial x_j} = \eta \frac{\partial^2 B_i}{\partial x_j^2},$$

where  $D_{ij} = u_i u_j - B_i B_j$ ,  $S_{ij} = \delta_{ij} \text{tr } D/3$ , and

$$G_{ij} = B_i u_j - u_i B_j.$$

- The traceless matrix  $D_{ij} - S_{ij}$  has 8 independent components.
- The antisymmetric matrix  $G_{ij}$  has only 3.
- An additional 6 FFT calls are required to compute the components of  $\mathbf{u}$  and  $\mathbf{B}$  in  $x$  space.
- The MHD nonlinearity can thus be computed with 17 FFT calls.

# Discrete Cyclic Convolution

- The FFT provides an efficient tool for computing the *discrete cyclic convolution*

$$\sum_{p=0}^{N-1} F_p G_{k-p},$$

where the vectors  $F$  and  $G$  have period  $N$ .

- The backward 1D *discrete Fourier transform* of a complex vector  $\{F_k : k = 0, \dots, N - 1\}$  is defined as

$$f_j \doteq \sum_{k=0}^{N-1} \zeta_N^{jk} F_k, \quad j = 0, \dots, N - 1,$$

where  $\zeta_N = e^{2\pi i/N}$  denotes the  *$N$ th primitive root of unity*.

- The fast Fourier transform (FFT) method exploits the properties that  $\zeta_N^r = \zeta_{N/r}$  and  $\zeta_N^N = 1$ .

# Convolution Theorem

$$\begin{aligned}
 \sum_{j=0}^{N-1} f_j g_j \zeta_N^{-jk} &= \sum_{j=0}^{N-1} \zeta_N^{-jk} \left( \sum_{p=0}^{N-1} \zeta_N^{jp} F_p \right) \left( \sum_{q=0}^{N-1} \zeta_N^{jq} G_q \right) \\
 &= \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} F_p G_q \sum_{j=0}^{N-1} \zeta_N^{(-k+p+q)j} \\
 &= N \sum_s \sum_{p=0}^{N-1} F_p G_{k-p+sN}.
 \end{aligned}$$

- The terms indexed by  $s \neq 0$  are *aliases*; we need to remove them!
- If only the first  $m$  entries of the input vectors are nonzero, aliases can be avoided by *zero padding* input data vectors of length  $m$  to length  $N \geq 2m - 1$ .
- *Explicit zero padding* prevents mode  $m - 1$  from beating with itself, wrapping around to contaminate mode  $N = 0 \pmod N$ .

# Implicit Dealiasing

- Let  $N = 2m$ . For  $j = 0, \dots, 2m - 1$  we want to compute

$$f_j = \sum_{k=0}^{2m-1} \zeta_{2m}^{jk} F_k.$$

- If  $F_k = 0$  for  $k \geq m$ , one can easily avoid looping over the unwanted zero Fourier modes by decimating in wavenumber:

$$f_{2\ell} = \sum_{k=0}^{m-1} \zeta_{2m}^{2\ell k} F_k = \sum_{k=0}^{m-1} \zeta_m^{\ell k} F_k,$$

$$f_{2\ell+1} = \sum_{k=0}^{m-1} \zeta_{2m}^{(2\ell+1)k} F_k = \sum_{k=0}^{m-1} \zeta_m^{\ell k} \zeta_{2m}^k F_k, \quad \ell = 0, 1, \dots, m - 1.$$

- This requires computing two subtransforms, each of size  $m$ , for an overall computational scaling of order  $2m \log_2 m = N \log_2 m$ .

- Parallelized multidimensional implicit dealiasing routines have been implemented as a software layer **FFTW++** (v2.05) on top of the **FFTW** library under the Lesser GNU Public License:

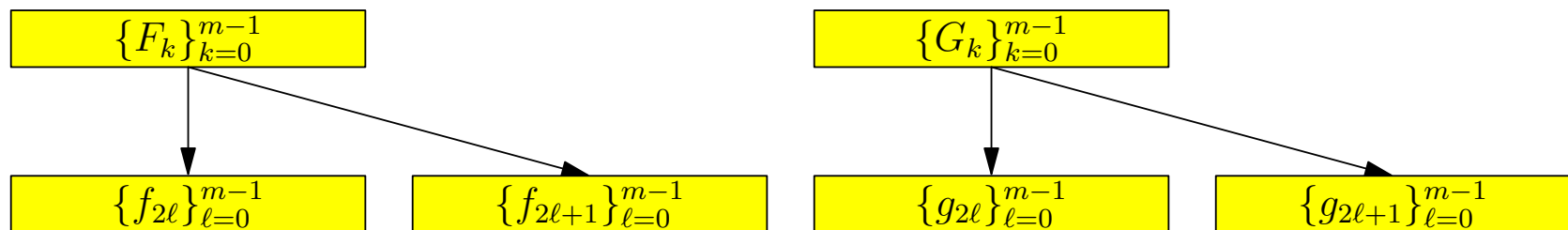
<http://fftwpp.sourceforge.net/>

$$\{F_k\}_{k=0}^{m-1}$$

$$\{G_k\}_{k=0}^{m-1}$$

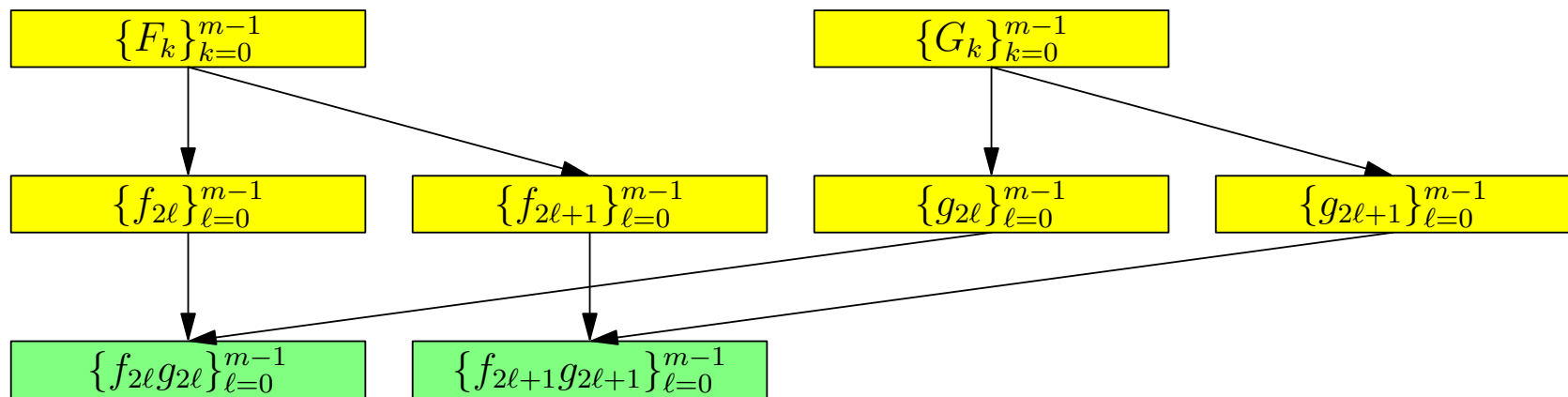
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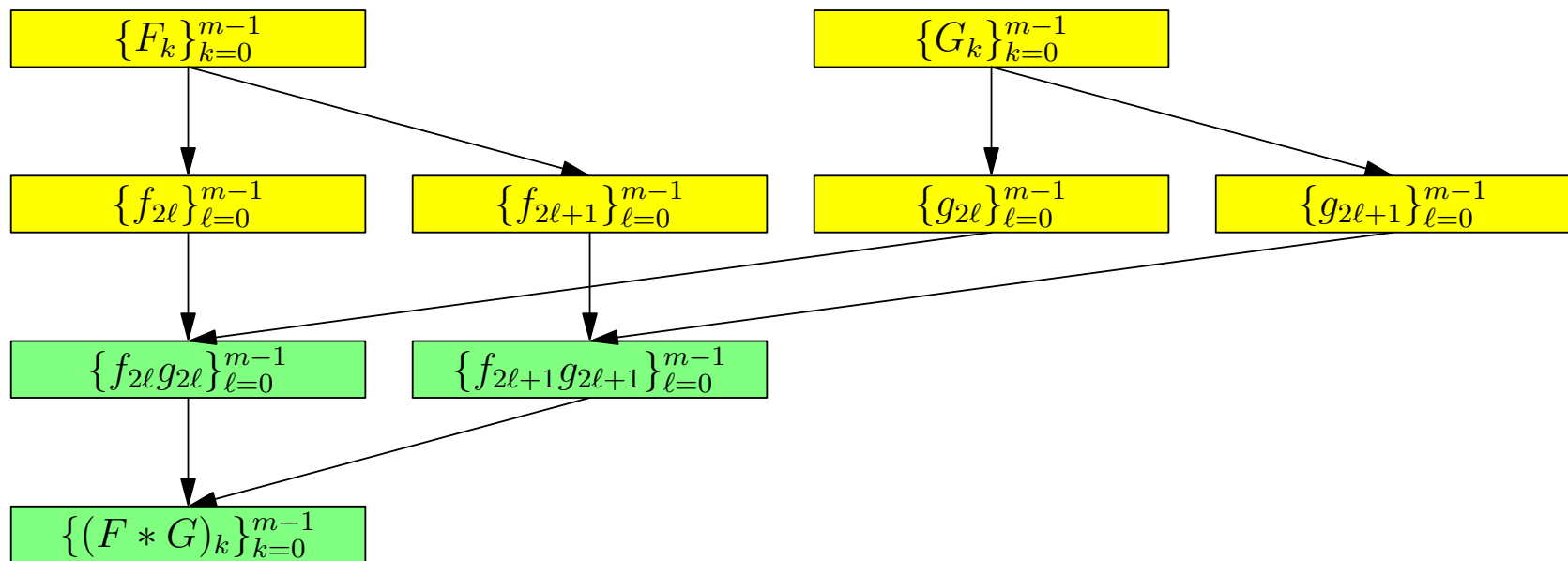
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# Conclusions

- The upper bound in the  $Z-E$  plane obtained previously for constant forcing also works for white-noise forcing.
- Adding **hypoviscosity** to the Navier–Stokes equation has a **dramatic effect on the turbulent dynamics**: it restricts the global attractor to the region characterized by the forcing annulus.
- With these tools, it should now be possible to study the relation between white-noise and constant forcings by examining their effects on the global attractor.
- This may lead to an explicit relation for the energy and enstrophy injection rates for constant forcing.
- Analytical bounds for random forcing provide a means to evaluate various heuristic turbulent subgrid (and supergrid!) models by characterizing the behaviour of the global attractor under these models.

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