Mode Reduction Schemes and Subgrid Models for Homogeneous Turbulence

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Outline

2D Turbulence

- 2D Navier–Stokes vorticity equation:

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} = \int d\boldsymbol{p} \int d\boldsymbol{q}\, \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}},$$

  where $\nu_{\boldsymbol{k}} \doteq \nu k^2$ and

$$\epsilon_{\boldsymbol{kpq}} \doteq (\hat{\boldsymbol{z}} \cdot \boldsymbol{p} \times \boldsymbol{q})\, \delta(\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q})$$

  is antisymmetric under permutation of any two indices.

- Energy E_0 and enstrophy Z_0 on the fine grid:

$$E_0 \doteq \frac{1}{2} \int d\boldsymbol{k}\, \frac{|\omega_{\boldsymbol{k}}|^2}{k^2}, \qquad Z_0 \doteq \frac{1}{2} \int d\boldsymbol{k}\, |\omega_{\boldsymbol{k}}|^2 .$$

- First consider $\nu_{\boldsymbol{k}} = 0$. Conservation of E_0 and Z_0 follow from:

$$\frac{1}{k^2} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \quad \text{antisymmetric in} \quad \boldsymbol{k} \leftrightarrow \boldsymbol{q},$$

$$\frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \quad \text{antisymmetric in} \quad \boldsymbol{k} \leftrightarrow \boldsymbol{p}.$$

Spectral Reduction

- Introduce a coarse-grained grid indexed by K.
- Define new variables

$$\Omega_{\boldsymbol{K}} = \langle \omega_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \doteq \frac{1}{\Delta_{\boldsymbol{K}}} \int_{\Delta_K} \omega_{\boldsymbol{k}} \, d\boldsymbol{k},$$

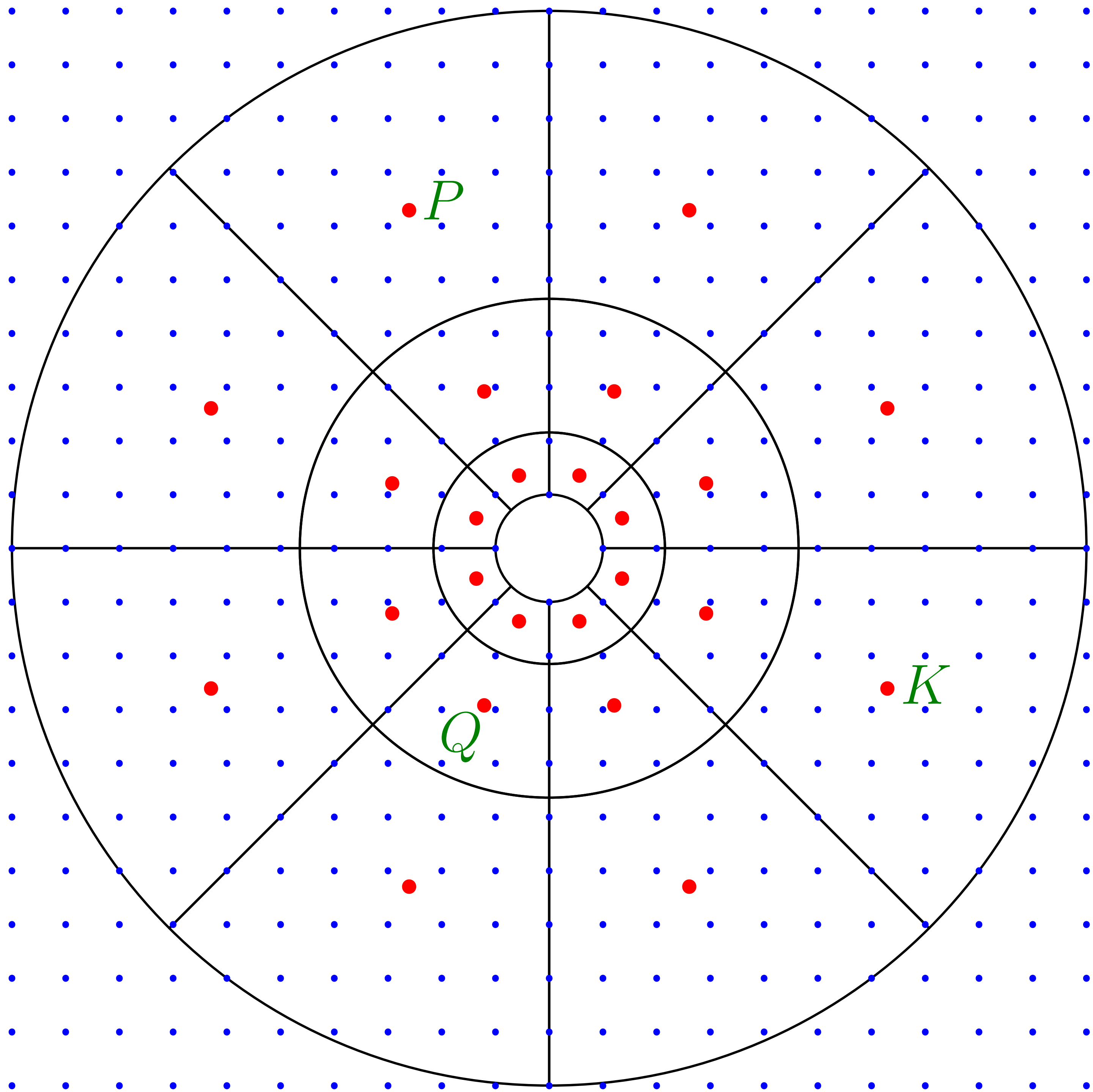
  where $\Delta_{\boldsymbol{K}}$ is the area of bin $\boldsymbol{K}$.
- Evolution of $\Omega_{\boldsymbol{K}}$:

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^* \right\rangle_{\boldsymbol{KPQ}},$$

  where $\langle f \rangle_{\boldsymbol{KPQ}} = \dfrac{1}{\Delta_{\boldsymbol{K}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}}} \displaystyle\int_{\Delta_K} d\boldsymbol{k} \int_{\Delta_P} d\boldsymbol{p} \int_{\Delta_Q} d\boldsymbol{q} \, f.$
- Approximate $\omega_{\boldsymbol{p}}$ and $\omega_{\boldsymbol{q}}$ by bin-averaged values $\Omega_{\boldsymbol{P}}$ and $\Omega_{\boldsymbol{Q}}$:

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \Omega_{\boldsymbol{P}}^* \Omega_{\boldsymbol{Q}}^*.$$

Wavenumber Bin Geometry (3×8 bins)



- On the coarse grid, define the energy E and enstrophy Z

$$E \doteq \frac{1}{2} \sum_{\boldsymbol{K}} \frac{|\Omega_{\boldsymbol{K}}|^2}{K^2} \Delta_{\boldsymbol{K}}, \qquad Z \doteq \frac{1}{2} \sum_{\boldsymbol{K}} |\Omega_{\boldsymbol{K}}|^2 \Delta_{\boldsymbol{K}}.$$

- Enstrophy is still conserved since

$$\left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \qquad \text{antisymmetric in} \qquad \boldsymbol{K} \leftrightarrow \boldsymbol{P}.$$

- But energy conservation has been lost!

$$\frac{1}{K^2} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \qquad \text{NOT antisymmetric in} \qquad \boldsymbol{K} \leftrightarrow \boldsymbol{Q}.$$

- Reinstate both desired symmetries with the modified coefficient

$$\frac{\left\langle \epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{KPQ}}}{Q^2}.$$

- Energy and enstrophy are now simultaneously conserved.

Properties

- We call the forced-dissipative version of this approximation *Spectral Reduction* (SR):

$$\boxed{\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \, \Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \, \frac{\left\langle \epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{KPQ}}}{Q^2} \, \Omega^*_{\boldsymbol{P}} \Omega^*_{\boldsymbol{Q}}.}$$

- SR conserves both energy and enstrophy and reduces to the exact dynamics in the limit of small bin size.
- It has the same general structure and symmetries as the original equation and in this sense may be considered a *renormalization*.
- SR obeys a Liouville Theorem; in the inviscid limit, it yields statistical-mechanical (equipartition) solutions.

Moments

- Q. How accurate is Spectral Reduction?
- A. For large bins, the *instantaneous* dynamics of SR is inaccurate.
- However: the equations for the *time-averaged* (or ensemble-averaged) moments predicted by SR closely approximate those of the exact bin-averaged statistics.
  Eg., time average the exact bin-averaged enstrophy equation:

$$\overline{\frac{\partial}{\partial t}\left\langle \left|\omega_{\boldsymbol{k}}\right|^2\right\rangle_{\boldsymbol{K}}} + 2\,\mathrm{Re}\left\langle \nu_{\boldsymbol{k}}\overline{\left|\omega_{\boldsymbol{k}}\right|^2}\right\rangle_{\boldsymbol{K}} = 2\,\mathrm{Re}\sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}}\Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2}\overline{\omega^*_{\boldsymbol{k}}\omega^*_{\boldsymbol{p}}\omega^*_{\boldsymbol{q}}}\right\rangle_{\boldsymbol{KPQ}}$$

  where the bar means time average and $\langle\cdot\rangle_{\boldsymbol{K}}$ means bin average.

- Time-averaged quantities such as $\overline{\left|\omega_{\boldsymbol{k}}\right|^2}$ and $\overline{\omega^*_{\boldsymbol{k}}\omega^*_{\boldsymbol{p}}\omega^*_{\boldsymbol{q}}}$ are generally *smooth* functions of $\boldsymbol{k}$, $\boldsymbol{p}$, $\boldsymbol{q}$ on the four-dimensional surface defined by the triad condition $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0$.

- Mean Value Theorem for integrals: for some $\boldsymbol{\xi} \in \boldsymbol{K}$,

$$\overline{|\Omega_{\boldsymbol{K}}|^2} = \overline{|\omega_{\boldsymbol{\xi}}|^2} \approx \overline{|\omega_{\boldsymbol{k}}|^2} \qquad \forall \boldsymbol{k} \in \boldsymbol{K}.$$

- To good accuracy these statistical moments may therefore be evaluated at the characteristic wavenumbers $\boldsymbol{K}$, $\boldsymbol{P}$, $\boldsymbol{Q}$:

$$\overline{\frac{\partial}{\partial t} |\Omega_{\boldsymbol{K}}|^2} + 2\,\mathrm{Re}\,\langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \overline{|\Omega_{\boldsymbol{K}}|^2} = 2\,\mathrm{Re} \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \overline{\Omega^*_{\boldsymbol{K}} \Omega^*_{\boldsymbol{P}} \Omega^*_{\boldsymbol{Q}}}.$$

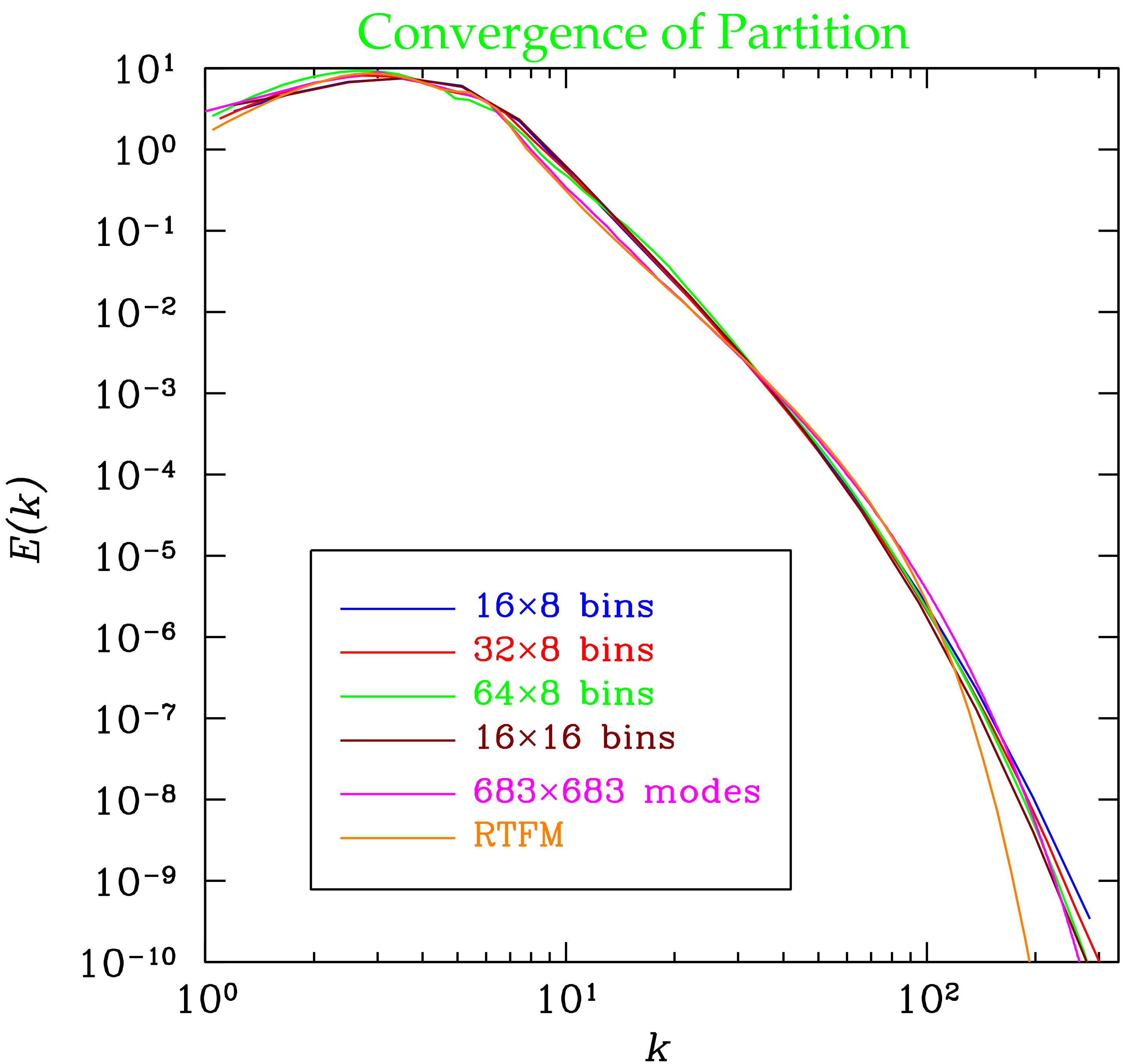
To the extent that the wavenumber magnitude q varies slowly over a bin:

$$\overline{\frac{\partial}{\partial t} |\Omega_{\boldsymbol{K}}|^2} + 2\,\mathrm{Re}\,\langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \overline{|\Omega_{\boldsymbol{K}}|^2} = 2\,\mathrm{Re} \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \frac{\langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}}}{Q^2} \overline{\Omega^*_{\boldsymbol{K}} \Omega^*_{\boldsymbol{P}} \Omega^*_{\boldsymbol{Q}}}.$$

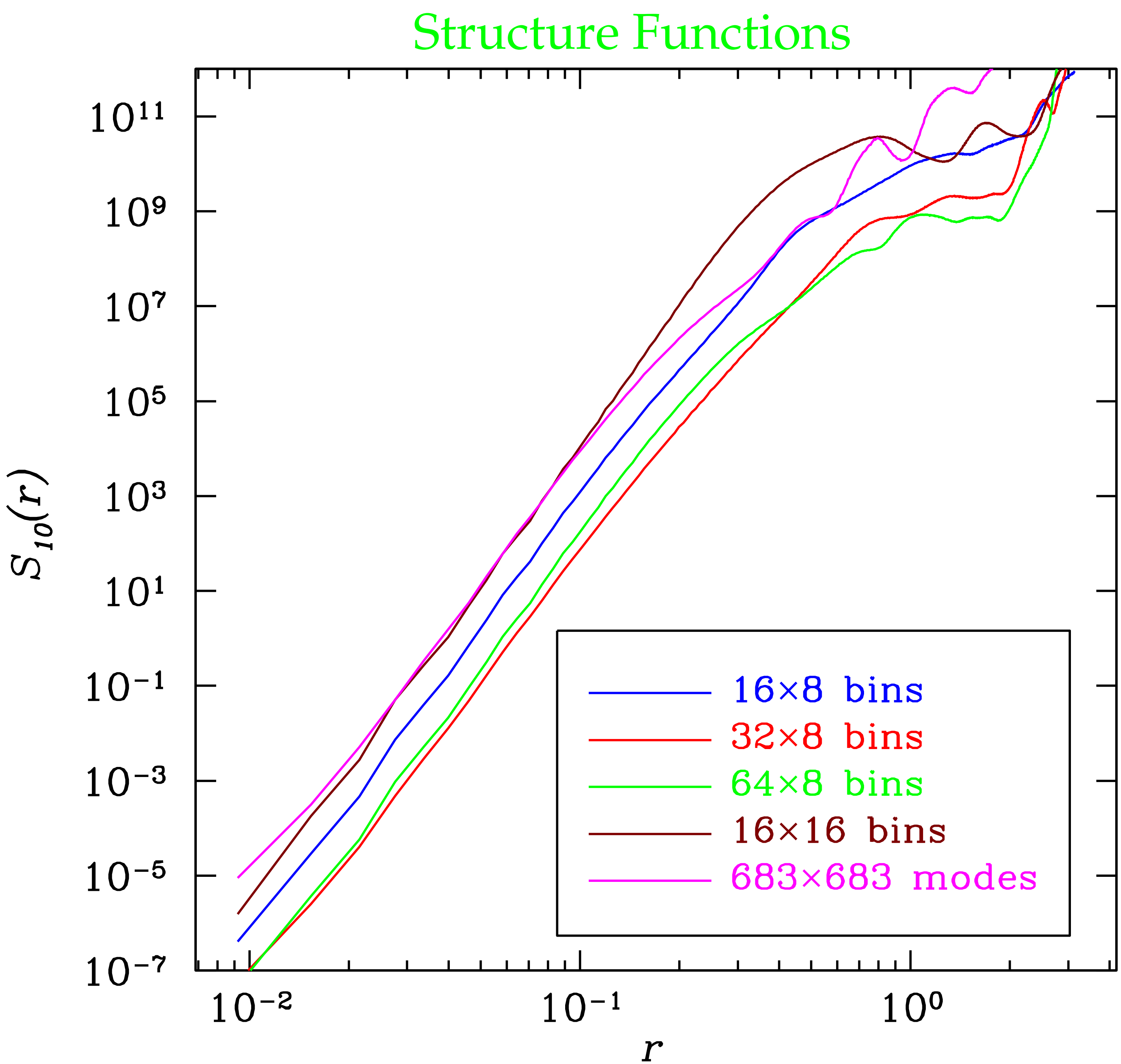
- But this is precisely the time-average of the SR equation!

Convergence

- The previous argument suggests that Spectral Reduction can indeed provide an accurate statistical description of turbulence, even when each bin contains many statistically independent modes.
- As the wavenumber partition is refined, one expects the solutions of the time-averaged SR equations to converge to the exact statistical solution.
- An object-oriented C^{++} program `(Triad)` has been developed to implement and test Spectral Reduction.



Convergence of Partition
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
$E(k)$
16×8 bins
32×8 bins
64×8 bins
16×16 bins
683×683 modes
RTFM
10^{0}
10^{1}
10^{2}
k



Structure Functions
10^{11}
10^{9}
10^{7}
10^{5}
10^{3}
10^{1}
10^{-1}
10^{-3}
10^{-5}
10^{-7}
$S_{10}(r)$
10^{-2}
10^{-1}
10^{0}
r
16×8 bins
32×8 bins
64×8 bins
16×16 bins
683×683 modes

Noncanonical Hamiltonian Formulation

- Underlying *noncanonical* Hamiltonian formulation for inviscid 2D vorticity equation:

$$\dot{\omega}_{\boldsymbol{k}} = \int d\boldsymbol{q}\, J_{\boldsymbol{k}\boldsymbol{q}} \frac{\delta H}{\delta \omega_{\boldsymbol{q}}},$$

where

$$H \doteq \frac{1}{2} \int d\boldsymbol{k}\, \frac{|\omega_{\boldsymbol{k}}|^2}{k^2},$$

$$J_{\boldsymbol{k}\boldsymbol{q}} \doteq \int d\boldsymbol{p}\, \epsilon_{\boldsymbol{k}\boldsymbol{p}\boldsymbol{q}} \omega^*_{\boldsymbol{p}}.$$

- Leads to inviscid Navier–Stokes equation:

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} = \int d\boldsymbol{p} \int d\boldsymbol{q}\, \frac{\epsilon_{\boldsymbol{k}\boldsymbol{p}\boldsymbol{q}}}{q^2} \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}}.$$

Liouville Theorem

- Navier–Stokes:

$$J_{\boldsymbol{kq}} \doteq \int d\boldsymbol{p}\, \epsilon_{\boldsymbol{kpq}} \omega^*_{\boldsymbol{p}}$$

$$\Rightarrow \qquad \int d\boldsymbol{k}\, \frac{\delta \dot{\omega}_{\boldsymbol{k}}}{\delta \omega_{\boldsymbol{k}}} = \int d\boldsymbol{k} \int d\boldsymbol{q}\, \underbrace{\frac{\delta J_{\boldsymbol{kq}}}{\delta \omega_{\boldsymbol{k}}}}_{\epsilon_{\boldsymbol{k}(-\boldsymbol{k})\boldsymbol{q}} = 0} \frac{\delta H}{\delta \omega_{\boldsymbol{q}}} + J_{\boldsymbol{kq}} \frac{\delta^2 H}{\delta \omega_{\boldsymbol{k}} \delta \omega_{\boldsymbol{q}}} = 0.$$

- Spectral Reduction:

$$J_{\boldsymbol{KQ}} \doteq \sum_{\boldsymbol{P}} \Delta_{\boldsymbol{P}} \left\langle \epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{KPQ}} \Omega^*_{\boldsymbol{P}}$$

$$\Rightarrow \qquad \sum_{\boldsymbol{K}} \frac{\partial \dot{\Omega}_{\boldsymbol{K}}}{\partial \Omega_{\boldsymbol{K}}} = \sum_{\boldsymbol{K},\boldsymbol{Q}} \underbrace{\frac{\partial J_{\boldsymbol{KQ}}}{\partial \Omega_{\boldsymbol{K}}}}_{\left\langle \epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{K}(-\boldsymbol{K})\boldsymbol{Q}} = 0} \frac{\partial H}{\partial \Omega_{\boldsymbol{Q}}} + J_{\boldsymbol{KQ}} \frac{\partial^2 H}{\partial \Omega_{\boldsymbol{K}} \partial \Omega_{\boldsymbol{Q}}} = 0.$$

Statistical Equipartition

- If the dynamics are *mixing*, the Liouville Theorem and the coarse-grained invariants

$$E \doteq \frac{1}{2}\sum_{\boldsymbol{K}} \frac{|\Omega_{\boldsymbol{K}}|^2}{K^2}\Delta_{\boldsymbol{K}}, \qquad Z \doteq \frac{1}{2}\sum_{\boldsymbol{K}} |\Omega_{\boldsymbol{K}}|^2\,\Delta_{\boldsymbol{K}},$$

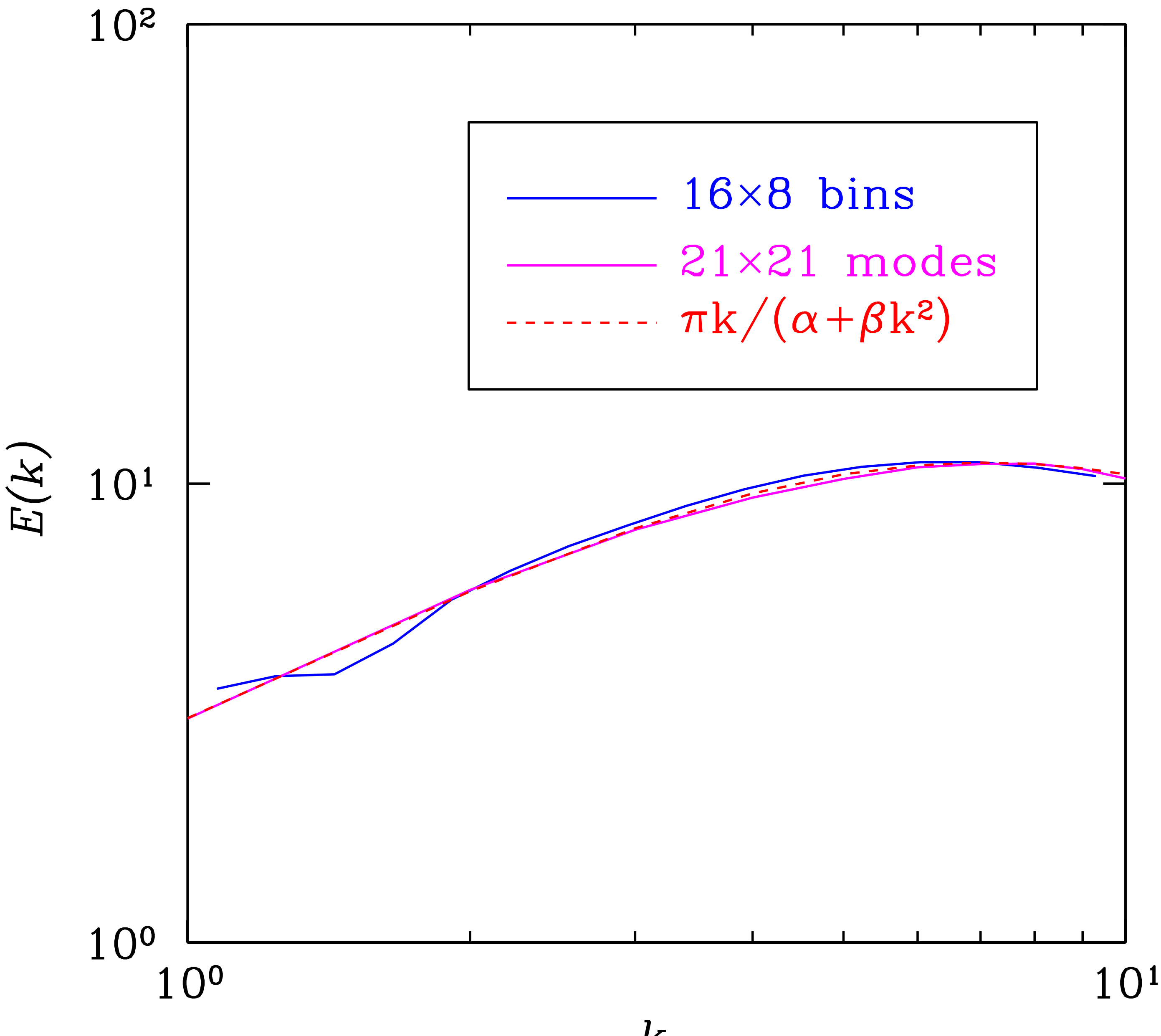
  lead to statistical equipartition of $(\alpha/K^2+\beta)\,|\Omega_{\boldsymbol{K}}|^2\,\Delta_{\boldsymbol{K}}$.

- This is the correct equipartition only for uniform bins. However, for nonuniform bins, a rescaling of time by $\Delta_{\boldsymbol{K}}$:

$$\boxed{\frac{1}{\Delta_{\boldsymbol{K}}}\frac{\partial\Omega_{\boldsymbol{K}}}{\partial t} + \langle\nu_{\boldsymbol{k}}\rangle_{\boldsymbol{K}}\,\Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}}\Delta_{\boldsymbol{Q}}\,\frac{\langle\epsilon_{\boldsymbol{kpq}}\rangle_{\boldsymbol{KPQ}}}{Q^2}\,\Omega^*_{\boldsymbol{P}}\Omega^*_{\boldsymbol{Q}}.}$$

  yields the correct inviscid equipartition:

$$\left\langle|\Omega_{\boldsymbol{k}}|^2\right\rangle = \frac{1}{\frac{\alpha}{K^2}+\beta}.$$



Relaxation to equipartition

Stiffness Problem

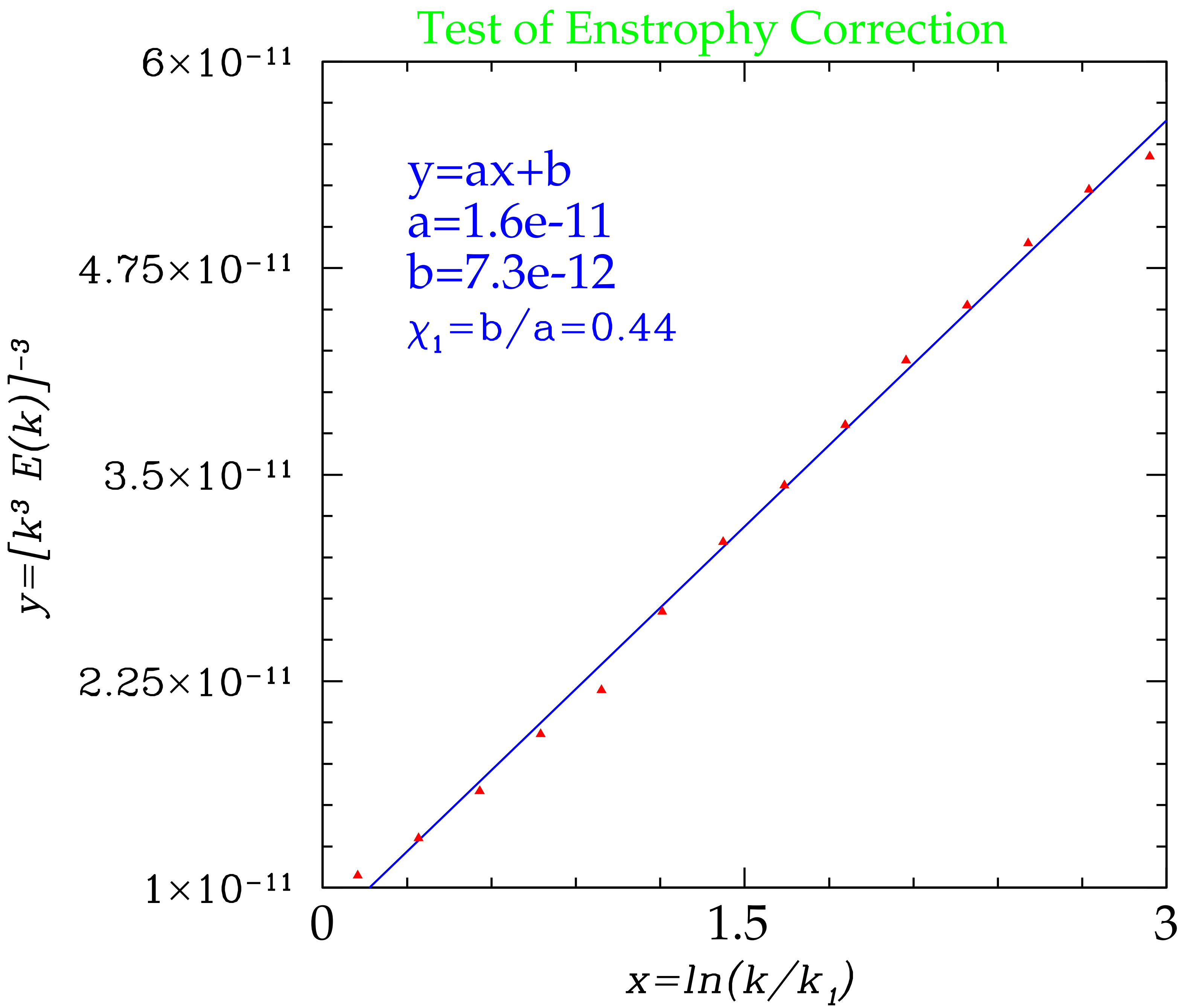
- The rescaling of time does not change the steady-state moment equations.
- It does affect the statistical trajectory of the system and the resulting statistical solution.
- However, the resulting system becomes numerically very stiff.
- Unsolved Problem: given an efficient numerical method for evolving the system of equations

$$\frac{d\boldsymbol{y}}{dt} = \boldsymbol{S}(\boldsymbol{y}),$$

find an efficient numerical method to evolve

$$\frac{d\boldsymbol{y}}{dt} = \boldsymbol{\Lambda}\boldsymbol{S}(\boldsymbol{y}),$$

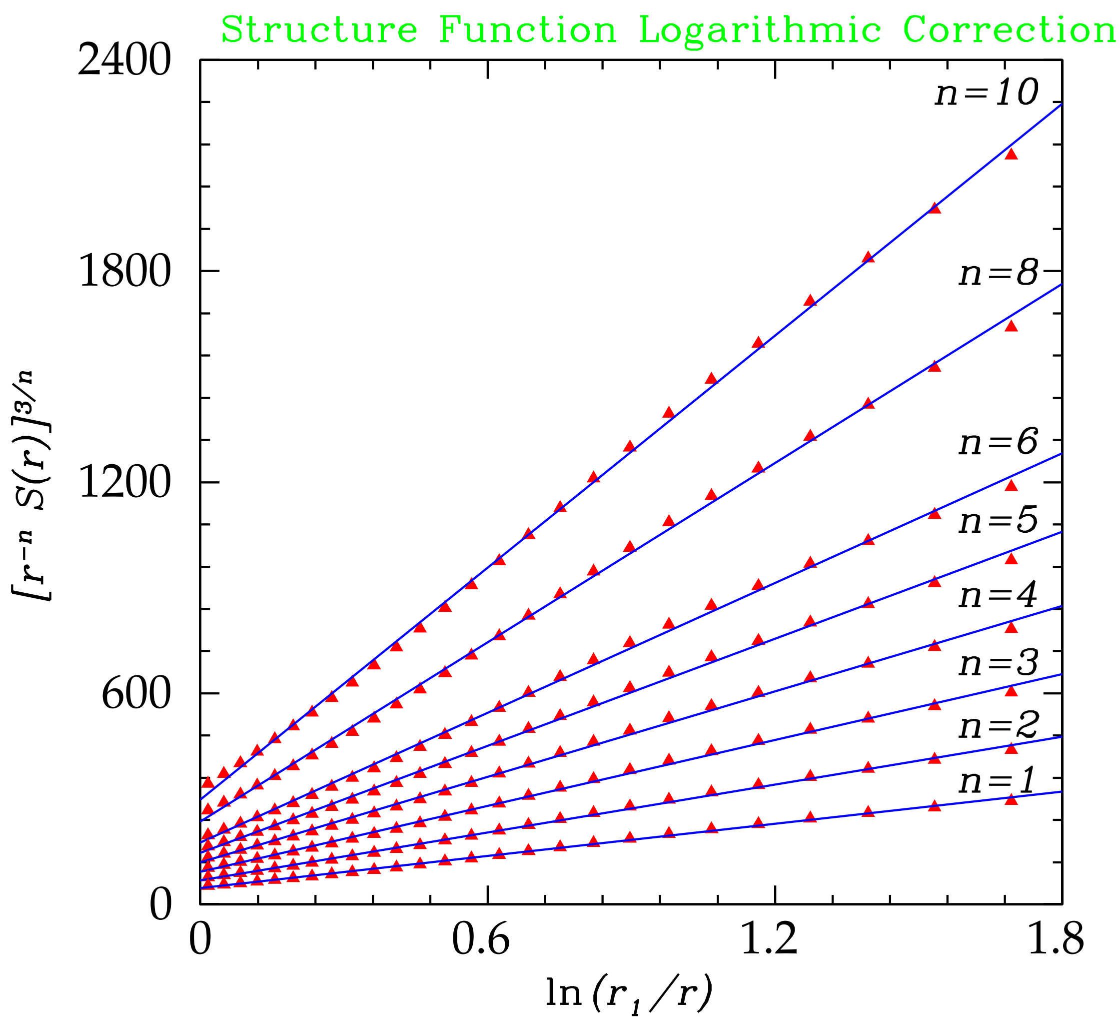
where $\boldsymbol{\Lambda}$ is a constant real diagonal matrix.


Test of Enstrophy Correction
y=ax+b
a=1.6e-11
b=7.3e-12
χ_1=b/a=0.44
6×10^{-11}
4.75×10^{-11}
3.5×10^{-11}
2.25×10^{-11}
1×10^{-11}
0
1.5
3
$x=ln(k/k_1)$
$y=[k^3\ E(k)]^{-3}$

Structure functions:

- [Falkovich & Lebedev 1994], [Paret *et al.* 1999]

$$S_n(\boldsymbol{r}) \doteq \overline{|v(\boldsymbol{r}) - v(\mathbf{0})|^n} \sim r^n \left[\log\left(\frac{r_1}{r}\right) + \chi'_n\right]^{n/3}.$$



GOY Shell Model

- Complex version of the Gledzer [1973] model proposed by Yamada and Ohkitani [1987]:

$$\left(\frac{d}{dt}+\nu k_n^2\right)u_n = ik_n\left(\alpha u_{n+1}^* u_{n+2}^* + \frac{\beta}{\lambda}u_{n-1}^* u_{n+1}^* + \frac{\gamma}{\lambda^2}u_{n-1}^* u_{n-2}^*\right) + F\delta_{n,0},$$

where

$$k_n = \lambda^n.$$

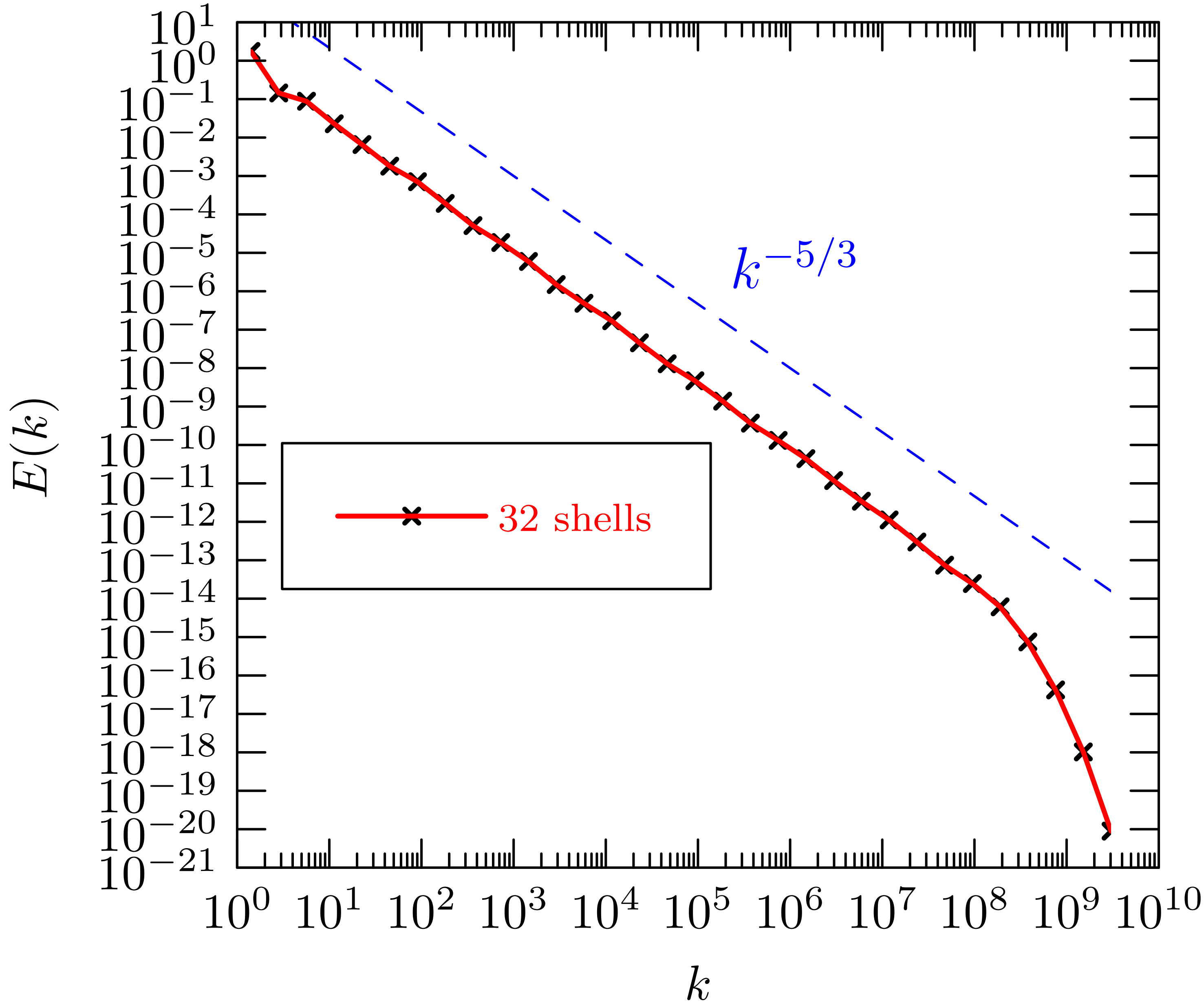
- With $\lambda = 2$, nonlinear terms conserve energy-like and helicity-like invariants

$$\alpha = 1 \qquad \beta = \gamma = -\frac{1}{2}.$$

- When $\nu = F = 0$, the GOY model has an unstable fixed point, corresponding to the Kolmogorov power law

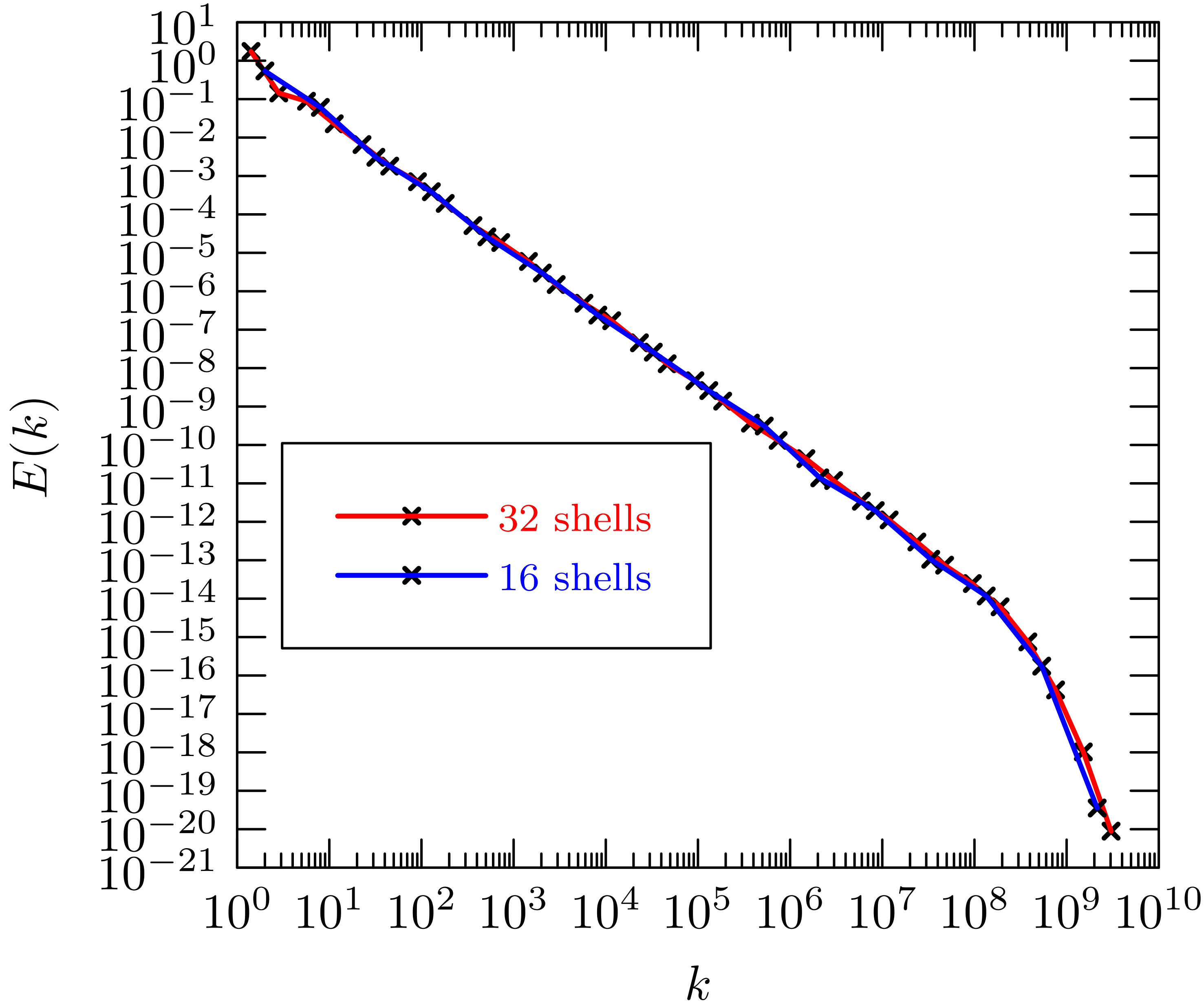
$$u_n = Ak_n^{-1/3}.$$

Kolmogorov Law



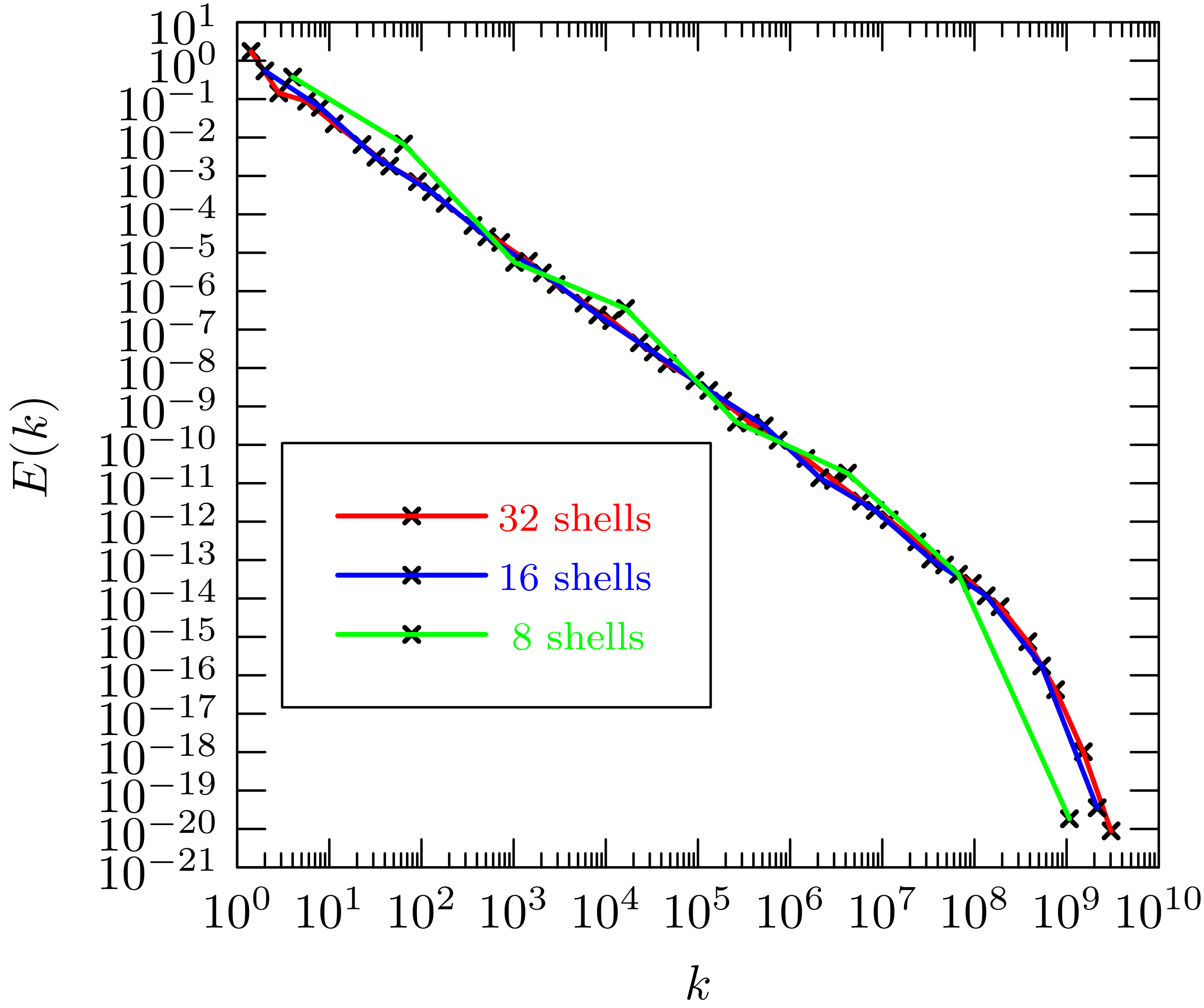
Energy spectrum for 3D GOY model

Spectral Reduction



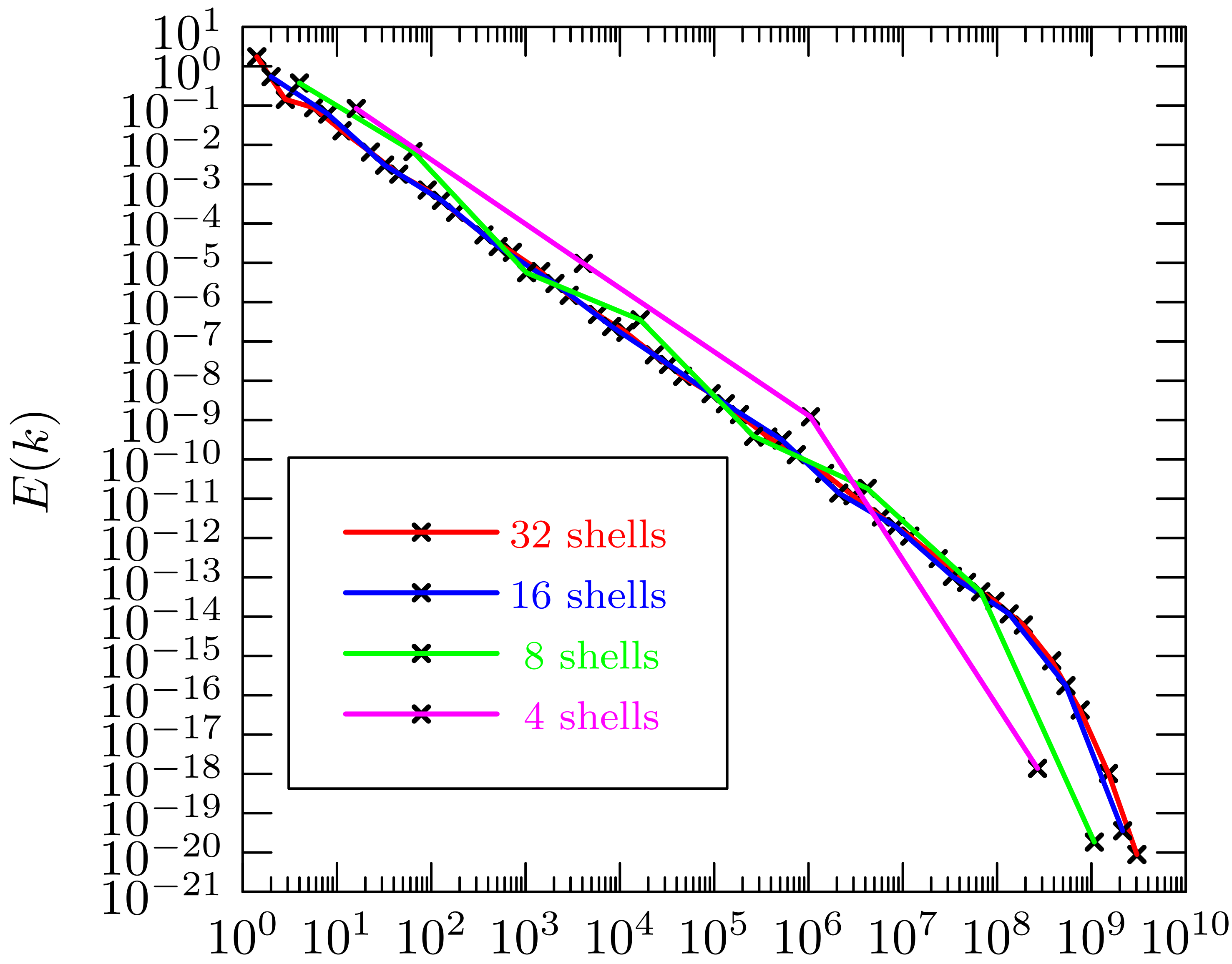
Energy spectrum for 3D GOY model

Spectral Reduction



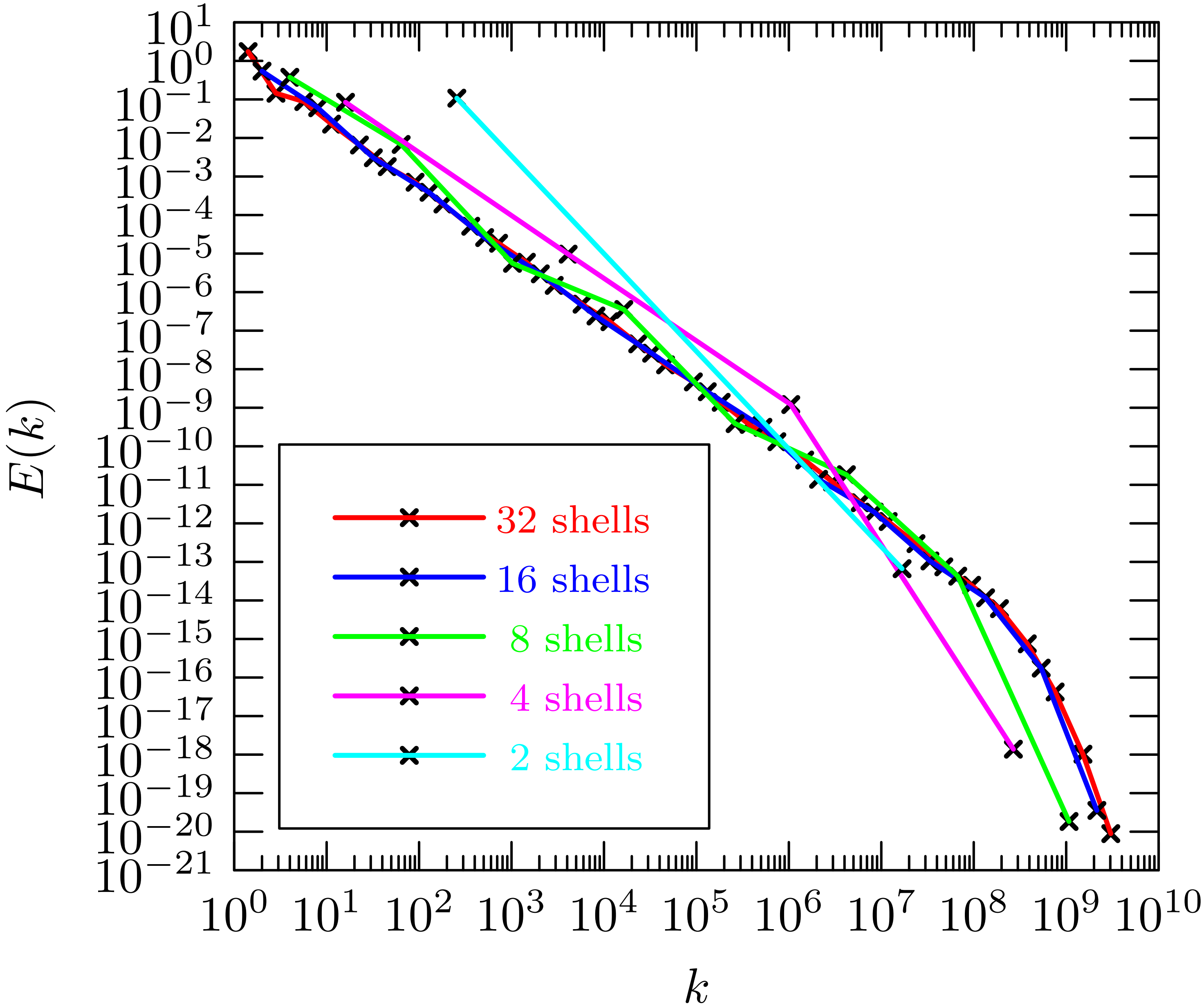
Energy spectrum for 3D GOY model

Spectral Reduction



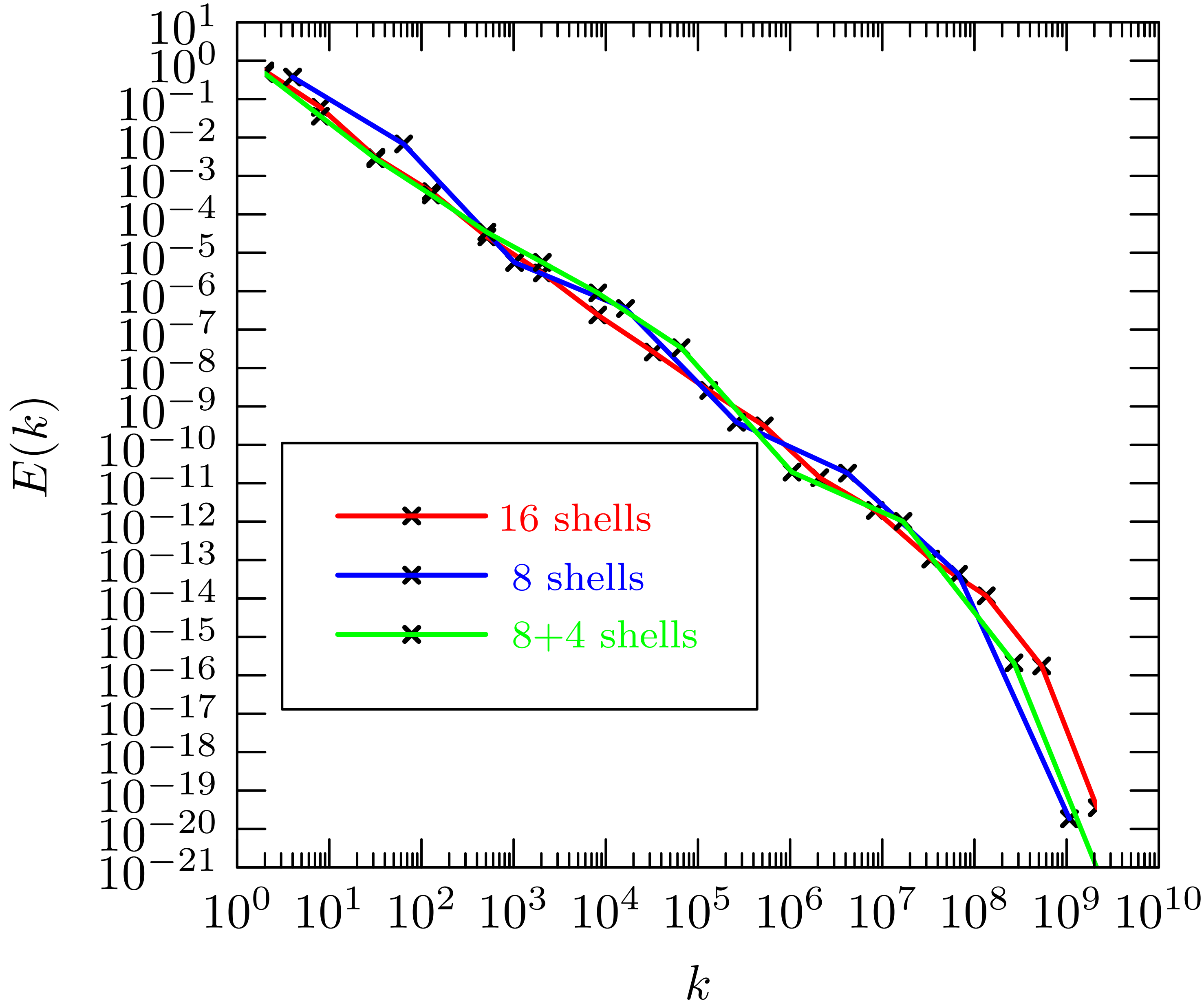
Energy spectrum for 3D GOY model

Spectral Reduction



Energy spectrum for 3D GOY model

Subgrid Model



Conclusions

- Spectral Reduction affords a dramatic reduction in the number of degrees of freedom that must be explicitly evolved in turbulence simulations.
- One can evolve a turbulent system for thousands of eddy turnover times to obtain energy spectra smooth enough to compare with theory.
- Spectral Reduction has been successfully applied to numerically verify the logarithmically corrected 2D enstrophy law to very high accuracy.
- The high-order structure functions computed by the pseudospectral method and Spectral Reduction are in excellent agreement at the small scales, even in the presence of coherent structures.
- Spectral Reduction lends numerical support to the theoretical and experimental claim that there are no intermittency corrections in strongly forced 2D enstrophy cascades.

Asymptote: The Vector Graphics Language

`http://asymptote.sf.net`

(freely available under the GNU public license)

References

[Bowman *et al.* 1999] J. C. Bowman, B. A. Shadwick, & P. J. Morrison, Phys. Rev. Lett., **83**:5491, 1999.

[Falkovich & Lebedev 1994] G. Falkovich & V. Lebedev, Phys. Rev. E, **50**:3883, 1994.

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