Bounds on the Global Attractor of 2D Incompressible Turbulence in the palenstrophy–enstrophy–energy space

John C. Bowman and Pedram Emami
Department of Mathematical and Statistical Sciences
University of Alberta

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Turbulence

• In 1941, Kolmogorov conjectured that the energy spectrum of 3D incompressible turbulence exhibits a self-similar power-law scaling characterized by a uniform \textit{cascade} of energy to molecular (viscous) scales:

\[ E(k) = C \epsilon^{2/3} k^{-5/3}. \]

• Here \( k \) is the Fourier wavenumber and \( E(k) \) is normalized so that \( \int E(k) \, dk \) is the total energy.

• Kolmogorov suggested that \( C \) might be a universal constant.
3D Energy Cascade

\[ \log E(k) \]

- **Forcing Range**
- **Inertial Range**
- **Dissipation Range**

\[ k^{−5/3} \]

\[ k_f \]

\[ k_d \]

\[ \log k \]
2D Energy Cascade

\[ \log E(k) \]

\[ \log k \]

Forcing Range

Inertial Range

Dissipation Range

\[ k^{-5/3} \]

\[ k^{-3} \]

\[ k_f \]

\[ k_d \]

\[ \log k \]
2D Turbulence

- Consider the Navier–Stokes equations for 2D incompressible homogeneous isotropic turbulence with density $\rho = 1$:

$$\frac{\partial u}{\partial t} - \nu \nabla^2 u + u \cdot \nabla u + \nabla P = F,$$

$$\nabla \cdot u = 0,$$

$$\int_{\Omega} u \, dx = 0, \quad \int_{\Omega} F \, dx = 0,$$

$$u(x, 0) = u_0(x),$$

with $\Omega = [0, 2\pi] \times [0, 2\pi]$ and periodic boundary conditions on $\partial \Omega$.

- Introduce the Hilbert space

$$H(\Omega) \doteq \text{cl} \left\{ u \in (C^2(\Omega) \cap L^2(\Omega))^2 \mid \nabla \cdot u = 0, \int_{\Omega} u \, dx = 0 \right\}.$$

with inner product $(u, v) = \int_{\Omega} u(x, t) \cdot v(x, t) \, dx$ and $L^2$ norm $|u| = (u, u)^{1/2}$.
• For \( u \in H(\Omega) \), the Navier–Stokes equations can be expressed:
\[
\frac{du}{dt} - \nu \nabla^2 u + u \cdot \nabla u + \nabla P = F.
\]

• Introduce \( A = -\mathcal{P}(\nabla^2) \), \( f = \mathcal{P}(F) \), and the bilinear map
\[
\mathcal{B}(u, u) = \mathcal{P}(u \cdot \nabla u + \nabla P),
\]
where \( \mathcal{P} : C^2(\Omega) \to H(\Omega) \) is the Helmholtz–Leray projection:
\[
\mathcal{P}(\nu) = \nu - \nabla \nabla^{-2} \nabla \cdot \nu.
\]

• The dynamical system can then be compactly written:
\[
\frac{du}{dt} + \nu A u + \mathcal{B}(u, u) = f.
\]
Stokes Operator $A$

- The operator $A = \mathcal{P}(-\nabla^2)$ is positive semi-definite and self-adjoint, with a compact inverse.

- On the periodic domain $\Omega = [0, 2\pi] \times [0, 2\pi]$, the eigenvalues of $A$ are
  \[ \lambda = \mathbf{k} \cdot \mathbf{k}, \quad \mathbf{k} \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\}. \]

- The eigenvalues of $A$ can be arranged as
  \[ 0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \quad \lambda_0 = 1 \]
  and its eigenvectors $\mathbf{w}_i, \ i \in \mathbb{N}_0$, form an orthonormal basis for the Hilbert space $H$, upon which we can define any quotient power of $A$:
  \[ A^\alpha \mathbf{w}_j = \lambda_j^\alpha \mathbf{w}_j, \quad \alpha \in \mathbb{R}, \ j \in \mathbb{N}_0. \]
Subspace of Finite Enstrophy

- We define the subspace of $H$ consisting of solutions with finite enstrophy:

$$V \doteq \left\{ u \in H \mid \sum_{j=0}^{\infty} \lambda_j(u, w_j)^2 < \infty \right\}.$$ 

- Another suitable norm for elements $u \in V$ is

$$\|u\| = \| A^{1/2}u \| = \left( \int_{\Omega} \left( \sum_{i=1}^{2} \frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_i} \right) \right)^{1/2} = \left( \sum_{j=0}^{\infty} \lambda_j(u, w_j)^2 \right)^{1/2}.$$
Quadratic Quantities

• For any solution $\boldsymbol{u}$ of the 2D Navier–Stokes equation, the $n$th-order quadratic quantity is

\[
E_n = \frac{1}{2} |A^n u|^2,
\]

• $E_0, \ Z \equiv E_{1/2}, \text{ and } P \equiv E_1$ are called the energy, enstrophy, and palinstrophy.
Properties of the Bilinear Map

• We make use of the antisymmetry

\[(B(u, v), w) = -(B(u, w), v),\]

which implies the conservation of the energy \( E_0 = \frac{1}{2}|u|^2. \)

• In 2D, we also have orthogonality:

\[(B(u, u), Au) = 0\]

and the strong form of enstrophy invariance:

\[(B(Av, v), u) = (B(u, v), Av).\]

which implies the conservation of the enstrophy \( E_\frac{1}{2} = \frac{1}{2}|A^{1/2}u|^2. \)

• In 2D, the above properties imply the symmetry

\[(B(v, v), Au) + (B(v, u), Av) + (B(u, v), Av) = 0.\]
Dynamical Behaviour

- Our starting point is the incompressible 2D Navier–Stokes equation with periodic boundary conditions:

\[
\frac{du}{dt} + \nu Au + B(u, u) = f, \quad u \in H.
\]

- Take the inner product with \( u \) (respectively \( Au \)):

\[
\frac{1}{2} \frac{d}{dt} |u(t)|^2 + \nu |u(t)|^2 = (f, u(t)),
\]

\[
\frac{1}{2} \frac{d}{dt} |u(t)|^2 + \nu |Au(t)|^2 = (f, Au(t)).
\]

- The Cauchy–Schwarz and Poincaré inequalities yield

\[(f, u(t)) \leq |f| |u(t)| \quad \text{and} \quad |u(t)| \leq |u(t)|.\]

- Since the existence and uniqueness for solutions to the 2D Navier–Stokes equation has been proven, a global attractor can be defined [Ladyzhenskaya 1975], [Foias & Temam 1979].
Dynamical Behaviour: Constant Forcing

- If the force $\mathbf{f}$ is constant with respect to time, a Gronwall inequality can be exploited:

$$|\mathbf{u}(t)|^2 \leq e^{-\nu t} |\mathbf{u}(0)|^2 + (1 - e^{-\nu t}) \left( \frac{|\mathbf{f}|}{\nu} \right)^2.$$  

- Defining a nondimensional Grashof number $G = \frac{|\mathbf{f}|}{\nu^2}$, the above inequality can be simplified to

$$|\mathbf{u}(t)|^2 \leq e^{-\nu t} |\mathbf{u}(0)|^2 + (1 - e^{-\nu t}) \nu^2 G^2.$$  

- Similarly,

$$||\mathbf{u}(t)||^2 \leq e^{-\nu t} ||\mathbf{u}(0)||^2 + (1 - e^{-\nu t}) \nu^2 G^2.$$  

- Being on the attractor thus requires

$$|\mathbf{u}| \leq \nu G \quad \text{and} \quad ||\mathbf{u}|| \leq \nu G.$$
Z–E Bounds: Constant Forcing

• A trivial lower bound is provided by the Poincaré inequality:

\[ |\mathbf{u}|^2 \leq ||\mathbf{u}||^2 \implies E \leq Z. \]

• An upper bound is given by

**Theorem 1 (Dascaliuc, Foias, and Jolly [2005])**

*For all \( \mathbf{u} \in \mathcal{A} \),

\[ ||\mathbf{u}||^2 \leq \frac{|\mathbf{f}|}{\nu} |\mathbf{u}|. \]

• That is,

\[ Z \leq \nu G \sqrt{E}. \]
Z–E Bounds: Constant Forcing
Extended Norm: Random Forcing

- For a random variable $\alpha$, with probability density function $P$, define the ensemble average

$$
\langle \alpha \rangle = \int_{-\infty}^{\infty} \alpha \left( \frac{dP}{d\zeta} \right) d\zeta.
$$

- The extended inner product is

$$
(u, v) \tilde{\omega} = \int_{\Omega} \langle u \cdot v \rangle d\mathbf{x} = \int_{\Omega} \left( \int_{-\infty}^{\infty} u \cdot v \frac{dP}{d\zeta} d\zeta \right) d\mathbf{x},
$$

with norm

$$
|f| \tilde{\omega} = \left( \int_{\Omega} \langle |f|^2 \rangle d\mathbf{x} \right)^{1/2}.
$$

- The $n$-th order injection rate is $\epsilon_n = (f, A^{2n}u)$. 
Dynamical Behaviour: Random Forcing

- Energy balance:
  \[ \frac{1}{2} \frac{d}{dt} |u|^2 + \nu (A u, u) + (B(u, u), u) = (f, u) \dot{=} \epsilon, \]
  where \( \epsilon \dot{=} \epsilon_0 \) is the rate of energy injection.

- From the energy conservation identity \((B(u, u), u) = 0\),
  \[ \frac{1}{2} \frac{d}{dt} |u|^2 + \nu |u|^2 = \epsilon. \]

- The Poincaré inequality \( ||u|| \geq |u| \) leads to
  \[ \frac{1}{2} \frac{d}{dt} |u|^2 \leq \epsilon - \nu |u|^2, \]
  which implies that \( |u(t)|^2 \leq e^{-2\nu t} |u(0)|^2 + \left( \frac{1 - e^{-2\nu t}}{\nu} \right) \epsilon. \]

- So for every \( u \in \mathcal{A}, \) we expect \( |u(t)|^2 \leq \epsilon/\nu. \)
• From $|u(t)| \leq \sqrt{\epsilon/\nu}$ we then obtain a lower bound for $|f|$: 

$$\sqrt{\nu \epsilon} \leq \frac{\epsilon}{|u|} = \frac{(f, u)}{|u|} \leq \frac{|f| |u|}{|u|} = |f|.$$ 

• It is convenient to use this lower bound for $|f|$ to define a lower bound for the Grashof number $G = |f|/\nu^2$, which we use as the normalization $\tilde{G}$ for random forcing:

$$\tilde{G} = \sqrt{\frac{\epsilon}{\nu^3}}.$$ 

• We proved the following theorem (JDE 2018):

**Theorem 2 (Emami & Bowman [2018])** *For all* $u \in A$ *with energy injection rate* $\epsilon$,*

$$||u||^2 \leq \sqrt{\frac{\epsilon}{\nu}} |u|.$$ 

• This leads to the same form as for a constant force: $Z \leq \nu \tilde{G} \sqrt{E}$. 
$Z - E$ Bounds: Random Forcing

\[
\frac{2Z}{\nu^2 G^2}
\]

\[
\frac{2E}{\nu^2 G^2}
\]
Z–E Bounds: Random Forcing

![Graph showing the relationship between 2Z/(νG²) and 2E/(νG²) over time (t). The graph includes a blue line and a red line representing different scales.](image-url)
3D Energy Spectrum
Large-Scale friction

• In the random-forcing case, we have recently extended the analysis to include a large-scale friction term:

\[
\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = -\nu_0 w + \nu \nabla^2 \omega + f.
\]

• If we generalize our definition of the Grashof number to account for \(\nu_0\):

\[
\tilde{G} = \sqrt{\epsilon (\nu + \nu_0)}
\]

\[
\tilde{G} = \frac{\sqrt{\epsilon (\nu + \nu_0)}}{\nu^2},
\]

the resulting analytic bounds retain the same form!
$Z - E$ Bounds: Random Forcing + Friction

\[
\frac{2Z}{(\nu G)^2} = 10^0, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}
\]
\[
\frac{2E}{(\nu G)^2} = 10^0, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}
\]
3D Energy Spectrum with Friction

\[ \frac{|u_k|^2}{2} \]
**P–Z Bounds**

- Just as the rate of energy dissipation is $2\nu Z$, the rate of enstropy dissipation is $2\nu P$ where $P$ is the palenstrophy.

- Dascaliuc, Foias, and Jolly also obtained bounds for the palenstrophy–enstrophy plane.

- A critical step in their argument is the application of the Cauchy–Schwarz inequality to estimate the bilinear triplet $(B(u, u), A^n u)$ for $n = 2$.

- For this bound to be sharp: $B(u, u) = \alpha A^n u$ a.e. for some $\alpha \in \mathbb{R}$.

- From the self-adjointness of $A$, such an alignment would require

\[
0 = (B(u, u), u) = (\alpha A^n u, u) = (\alpha A^{n/2} u, A^{n/2} u) = \alpha |A^{n/2} u|^2 \Rightarrow B(u, u) = 0 \text{ a.e.},
\]

which would imply no cascade!
• Numerical simulations show that these quantities are far from being aligned; in fact they are extremely close to being perpendicular!

• Consequently, the observed palenstrophy values are much lower than the predicted bounds.
$P - Z$ Upper Bounds

\[ P = 2\sqrt{\Lambda Z} \]

\[ P = (2CA^GZ)^2 \]

\[ P = \frac{(CA^GZ)^2}{2} \]

$P_1$ in here

\[ P = Z \]
$P-Z$ Bounds: Random Forcing + Friction
Isotropic turbulence

• For statistically isotropic turbulence, the expected value of the bilinear triplet \( \mathcal{B}(u, u), A^n u \) is zero:

**Theorem 3 (Emami & Bowman [2020])** In incompressible statistically isotropic 2D turbulence,

\[
\left\langle \int_{\Omega} (u \cdot \nabla) u \cdot A^n u \, dx \right\rangle = 0, \quad \forall n \in \mathbb{R}.
\]
• Proof: Express \( \mathbf{u} = (u, v) = (-\psi_y, \psi_x) \), where \( \psi \) is the stream function and define:

\[ \alpha \doteq -u_x = \psi_{yx} = v_y, \quad \beta \doteq -u_y = \psi_{yy}, \quad \gamma \doteq v_x = \psi_{xx}. \]

• Statistical isotropy then implies

\[
\left\langle \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot A^n \mathbf{u} \, d\mathbf{x} \right\rangle = \left\langle \int_\Omega (u u_x + v u_y) A^n u + (u v_x + v v_y) A^n v \, d\mathbf{x} \right\rangle \\
= \left\langle \int_\Omega (-\alpha u - \beta v) A^n u + (\gamma u + \alpha v) A^n v \, d\mathbf{x} \right\rangle \\
= \left\langle \int_\Omega \alpha (v A^n v - u A^n u) + (\gamma u A^n v - \beta v A^n u) \, d\mathbf{x} \right\rangle \\
= 0.
\]

• We then find, by normalizing to \( \tilde{G} = \sqrt{\frac{\epsilon_1}{2} (\nu + \nu_0) / \nu^2} \), that

\[
\frac{2P}{(\nu \tilde{G})^2} \leq \sqrt{\frac{\epsilon_1}{2} \frac{2Z}{(\nu \tilde{G})^2}}.
\]
General upper bound

• For every $\sigma \in \mathbb{R}$ and for all $\mathbf{u} \in \mathcal{A}$ driven by a random forcing having injection rate equal to $\epsilon_\sigma$,

**Theorem 4 (Emami & Bowman [2020])**

\[ |A^{\sigma+1/2} \mathbf{u}|^2 \leq \sqrt{\frac{\epsilon_\sigma}{\nu}} |A^\sigma \mathbf{u}|. \]
DNS code

- We have released a highly optimized 2D pseudospectral code in C++: https://github.com/dealiass/dns.

- It uses our FFTW++ library to implicitly dealias the advective convolution, while exploiting Hermitian symmetry [Bowman & Roberts 2011], [Roberts & Bowman 2018].

- Advanced computer memory management, such as implicit padding, memory alignment, and dynamic moment averaging allow DNS to attain its extreme performance.

- The formulation proposed by Basdevant [1983] is used to reduce the number of FFTs required for 2D (3D) incompressible turbulence to 4 (8).

- We also include simplified 2D (146 lines) and 3D (287 lines) versions called ProtoDNS for educational purposes: https://github.com/dealiass/dns/tree/master/protoDNS.
Implicit Dealiasing

• Let $N = 2m$. For $j = 0, \ldots, 2m - 1$ we want to compute

$$f_j = \sum_{k=0}^{2m-1} \zeta_{2m}^{jk} F_k.$$ 

• If $F_k = 0$ for $k \geq m$, one can easily avoid looping over the unwanted zero Fourier modes by decimating in wavenumber:

$$f_{2\ell} = \sum_{k=0}^{m-1} \zeta_{2m}^{2\ell k} F_k = \sum_{k=0}^{m-1} \zeta_m^{\ell k} F_k,$$

$$f_{2\ell + 1} = \sum_{k=0}^{m-1} \zeta_{2m}^{(2\ell + 1)k} F_k = \sum_{k=0}^{m-1} \zeta_m^{\ell k} \zeta_{2m}^k F_k, \quad \ell = 0, 1, \ldots, m - 1.$$ 

• This requires computing two subtransforms, each of size $m$, for an overall computational scaling of order $2m \log_2 m = N \log_2 m$. 
Parallelized multidimensional implicit dealiasing routines have been implemented as a software layer FFTW++ (v2.06) on top of the FFTW library under the Lesser GNU Public License:

http://fftwpp.sourceforge.net/
Conclusions

• The upper bound in the $Z-E$ plane obtained previously for constant forcing also works for white-noise forcing and large-scale friction (hypoviscosity).

• Previous bounds in the $P-Z$ plane vastly overestimate the values obtained from numerical simulations.

• These bounds can be greatly tightened by exploiting isotropy.

• Analytical bounds for random forcing provide a means to evaluate various heuristic turbulent subgrid (and supergrid!) models by characterizing the behaviour of the global attractor under these models.
References


