On the Global Attractor of 2D Incompressible Turbulence with Random Forcing and Friction

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Turbulence

Big whirls have little whirls that feed on their velocity, and little whirls have littler whirls and so on to viscosity... [Richardson 1922]

- In 1941, Kolmogorov conjectured that the energy spectrum of 3D incompressible turbulence exhibits a self-similar power-law scaling characterized by a uniform cascade of energy to molecular (viscous) scales:

\[ E(k) = C\epsilon^{2/3}k^{-5/3}. \]

- Here \( k \) is the Fourier wavenumber and \( E(k) \) is normalized so that \( \int E(k) \, dk \) is the total energy.

- Kolmogorov suggested that \( C \) might be a universal constant.
3D Energy Cascade

\[ \log E(k) \]

\[ \log k \]

Forcing Range
Inertial Range
Dissipation Range

\[ k^{-5/3} \]

\[ k_f \]

\[ k_d \]

\[ \log k \]
2D Incompressible Turbulence

- In 2D, where $\mathbf{u}$ maps a plane normal to $\hat{z}$ to $\mathbb{R}^2$, the vorticity vector $\omega = \nabla \times \mathbf{u}$ is always perpendicular to $\mathbf{u}$.

- Navier–Stokes equation for the scalar vorticity $\omega = \hat{z} \cdot \nabla \times \mathbf{u}$:
  \[
  \frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nu \nabla^2 \omega + f.
  \]

- The incompressibility condition $\nabla \cdot \mathbf{u} = 0$ can be exploited to find $\mathbf{u}$ in terms of $\omega$:
  \[
  \nabla \omega \times \hat{z} = \nabla \times \hat{z} \omega = \nabla \times (\nabla \times \mathbf{u}) = \nabla (\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u} = -\nabla^2 \mathbf{u}.
  \]

- Thus $\mathbf{u} = \hat{z} \times \nabla \nabla^{-2} \omega$. In Fourier space:
  \[
  \frac{d\omega_k}{dt} = S_k - \nu k^2 \omega_k + f_k,
  \]
  where $S_k = \sum_q \frac{\hat{z} \times q \cdot k}{q^2} \bar{\omega}_q \bar{\omega}_{-k-q} = \sum_{p,q} \frac{\epsilon_{kpq}}{q^2} \bar{\omega}_p \bar{\omega}_q$. 
Here $\epsilon_{kpq} \equiv \hat{z} \cdot p \times q \, \delta_{k+p+q}$ is antisymmetric under permutation of any two indices.

\[
\frac{d\omega_k}{dt} + \nu k^2 \omega_k = \sum_p \sum_q \frac{\epsilon_{kpq} q^2}{q^2} \omega_p \omega_q + f_k,
\]

- When $\nu = f_k = 0$:

\[
\text{enstrophy } Z = \frac{1}{2} \sum_k |\omega_k|^2 \text{ and energy } E = \frac{1}{2} \sum_k \frac{|\omega_k|^2}{k^2} \text{ are conserved:}
\]

\[
\frac{\epsilon_{kpq} q^2}{q^2} \text{ antisymmetric in } k \leftrightarrow p,
\]

\[
\frac{1}{k^2} \frac{\epsilon_{kpq} q^2}{q^2} \text{ antisymmetric in } k \leftrightarrow q.
\]
Fjørtoft Dual Cascade Scenario

\[ E_2 = E_1 + E_3, \quad Z_2 = Z_1 + Z_3, \quad Z_i \approx k_i^2 E_i. \]

- When \( k_1 = k, \) \( k_2 = 2k, \) and \( k_3 = 4k: \)

\[ E_1 \approx \frac{4}{5} E_2, \quad Z_1 \approx \frac{1}{5} Z_2, \quad E_3 \approx \frac{1}{5} E_2, \quad Z_3 \approx \frac{4}{5} Z_2. \]

- Fjørtoft [1953]: energy cascades to large scales and enstrophy cascades to small scales.
2D Energy Cascade

\[ \log E(k) \]

Forcing Range

Inertial Range

Dissipation Range

\[ k^{-3} \]

\[ k^{-5/3} \]

\[ k_f \]

\[ k_d \]

\[ \text{log } k \]
2D Turbulence: Mathematical Formulation

• Consider the Navier–Stokes equations for 2D incompressible homogeneous isotropic turbulence with density $\rho = 1$:

$$
\frac{\partial u}{\partial t} - \nu \nabla^2 u + u \cdot \nabla u + \nabla P = F,
$$

$$
\nabla \cdot u = 0,
$$

$$
\int_{\Omega} u \, dx = 0, \quad \int_{\Omega} F \, dx = 0,
$$

$$
u u(x, 0) = u_0(x),
$$

with $\Omega = [0, 2\pi] \times [0, 2\pi]$ and periodic boundary conditions on $\partial \Omega$.

• Introduce the Hilbert space

$$
H(\Omega) \doteq \text{cl} \left\{ u \in (C^2(\Omega) \cap L^2(\Omega))^2 \mid \nabla \cdot u = 0, \int_{\Omega} u \, dx = 0 \right\}.
$$

with inner product $\langle u, v \rangle = \int_{\Omega} u(x, t) \cdot v(x, t) \, dx$ and $L^2$ norm $|u| = (u, u)^{1/2}$. 
• For $u \in H(\Omega)$, the Navier–Stokes equations can be expressed:

$$\frac{du}{dt} - \nu \nabla^2 u + u \cdot \nabla u + \nabla P = F.$$ 

• Introduce $A \equiv -\mathcal{P}(\nabla^2)$, $f \equiv \mathcal{P}(F)$, and the bilinear map

$$\mathcal{B}(u, u) \equiv \mathcal{P}(u \cdot \nabla u + \nabla P),$$

where $\mathcal{P}$ is the Helmholtz–Leray projection operator from $(L^2(\Omega))^2$ to $H(\Omega)$:

$$\mathcal{P}(v) \equiv v - \nabla \nabla^{-2} \nabla \cdot v, \quad \forall v \in (L^2(\Omega))^2.$$

• The dynamical system can then be compactly written:

$$\frac{du}{dt} + \nu Au + \mathcal{B}(u, u) = f.$$
Stokes Operator $A$

- The operator $A = \mathcal{P}(-\nabla^2)$ is positive semi-definite and self-adjoint, with a compact inverse.

- On the periodic domain $\Omega = [0, 2\pi] \times [0, 2\pi]$, the eigenvalues of $A$ are

$$\lambda = k \cdot k, \quad k \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\}.$$

- The eigenvalues of $A$ can be arranged as

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \quad \lambda_0 = 1$$

and its eigenvectors $\mathbf{w}_i, \ i \in \mathbb{N}_0$, form an orthonormal basis for the Hilbert space $H$, upon which we can define powers of $A$:

$$A^\alpha \mathbf{w}_j = \lambda_j^\alpha \mathbf{w}_j, \quad \alpha \in \mathbb{R}, \quad j \in \mathbb{N}_0.$$
Subspace of Finite Enstrophy

- We define the subspace of $H$ consisting of solutions with finite enstrophy:

$$V = \left\{ u \in H \mid \sum_{j=0}^{\infty} \lambda_j(u, w_j)^2 < \infty \right\}.$$ 

- Another suitable norm for elements $u \in V$ is

$$||u|| = \left| A^{1/2} u \right| = \left( \int_{\Omega} \sum_{i=1}^{2} \frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_i} \right)^{1/2} = \left( \sum_{j=0}^{\infty} \lambda_j(u, w_j)^2 \right)^{1/2}.$$
Properties of the Bilinear Map

• We make use of the antisymmetry

\[(\mathcal{B}(u, v), w) = -(\mathcal{B}(u, w), v).\]

• In 2D, we also have orthogonality:

\[(\mathcal{B}(u, u), Au) = 0\]

and the strong form of enstrophy invariance:

\[(\mathcal{B}(Av, v), u) = (\mathcal{B}(u, v), Av).\]

• In 2D the above properties imply the symmetry

\[(\mathcal{B}(v, v), Au) + (\mathcal{B}(v, u), Av) + (\mathcal{B}(u, v), Av) = 0.\]
Dynamical Behaviour

- Our starting point is the incompressible 2D Navier–Stokes equation with periodic boundary conditions:

\[ \frac{d\mathbf{u}}{dt} + \nu \mathbf{Au} + \mathbf{B}(\mathbf{u}, \mathbf{u}) = \mathbf{f}, \quad \mathbf{u} \in H. \]

- Take the inner product with \( \mathbf{u} \) (respectively \( \mathbf{Au} \)):

\[ \frac{1}{2} \frac{d}{dt} |\mathbf{u}(t)|^2 + \nu |\mathbf{u}(t)|^2 = (\mathbf{f}, \mathbf{u}(t)), \]

\[ \frac{1}{2} \frac{d}{dt} |\mathbf{u}(t)|^2 + \nu |\mathbf{Au}(t)|^2 = (\mathbf{f}, \mathbf{Au}(t)). \]

- The Cauchy–Schwarz and Poincaré inequalities yield

\[ (\mathbf{f}, \mathbf{u}(t)) \leq |\mathbf{f}| |\mathbf{u}(t)| \quad \text{and} \quad |\mathbf{u}(t)| \leq |\mathbf{u}(t)|. \]

- Since the existence and uniqueness for solutions to the 2D Navier–Stokes equation has been proven, a global attractor can be defined [Ladyzhenskaya 1975], [Foias & Temam 1979].
Dynamical Behaviour: Constant Forcing

- If the force $f$ is constant with respect to time, a Gronwall inequality can be exploited:

$$|u(t)|^2 \leq e^{-\nu t} |u(0)|^2 + (1 - e^{-\nu t}) \left( \frac{|f|}{\nu} \right)^2.$$  

- Defining a nondimensional Grashof number $G = \frac{|f|}{\nu^2}$, the above inequality can be simplified to

$$|u(t)|^2 \leq e^{-\nu t} |u(0)|^2 + (1 - e^{-\nu t}) \nu^2 G^2.$$  

- Similarly,

$$||u(t)||^2 \leq e^{-\nu t} ||u(0)||^2 + (1 - e^{-\nu t}) \nu^2 G^2.$$  

- Being on the attractor thus requires

$$|u| \leq \nu G \quad \text{and} \quad ||u|| \leq \nu G.$$
Z–E Plane Bounds: Constant Forcing

• A trivial lower bound is provided by the Poincaré inequality:

\[ |u|^2 \leq ||u||^2 \implies E \leq Z. \]

• An upper bound is given by

Theorem 1 (Dascaliuc, Foias, and Jolly [2005])

For all \( u \in A \),

\[ ||u||^2 \leq \frac{|f|}{\nu} |u|. \]

• That is,

\[ 2Z \leq \nu G \sqrt{2E}. \]
$Z - E$ Plane Bounds: Constant Forcing

\[ \frac{2Z}{\nu^2 G^2} \]

\[ \frac{2E}{\nu^2 G^2} \]

A in here
Extended Norm: Random Forcing

• For a random variable $\alpha$, with probability density function $P$, define the ensemble average

$$\langle \alpha \rangle = \int_{-\infty}^{\infty} \alpha \left( \frac{dP}{d\zeta} \right) d\zeta.$$ 

• The extended inner product is

$$(u, v)_{\tilde{\omega}} = \int_{\Omega} \langle u \cdot v \rangle \, d\mathbf{x} = \int_{\Omega} \left( \int_{-\infty}^{\infty} u \cdot v \frac{dP}{d\zeta} d\zeta \right) d\mathbf{x},$$

with norm

$$|f|_{\tilde{\omega}} = \left( \int_{\Omega} \langle |f|^2 \rangle \, d\mathbf{x} \right)^{1/2}.$$
Dynamical Behaviour: Random Forcing

- Energy balance:
  \[
  \frac{1}{2} \frac{d}{dt} |u|^2 + \nu (Au, u) + (B(u, u), u) = (f, u) \equiv \epsilon,
  \]
  where \( \epsilon \) is the rate of energy injection.

- From the energy conservation identity \((B(u, u), u) = 0\),
  \[
  \frac{1}{2} \frac{d}{dt} |u|^2 + \nu |u|^2 = \epsilon.
  \]

- The Poincaré inequality \(||u|| \geq |u|\) leads to
  \[
  \frac{1}{2} \frac{d}{dt} |u|^2 \leq \epsilon - \nu |u|^2,
  \]
  which implies that
  \[
  |u(t)|^2 \leq e^{-2\nu t} |u(0)|^2 + \left( \frac{1 - e^{-2\nu t}}{\nu} \right) \epsilon.
  \]

- So for every \( u \in \mathcal{A} \), we expect \( |u(t)|^2 \leq \epsilon/\nu \).
• From $|u(t)| \leq \sqrt{\epsilon/\nu}$ we then obtain a lower bound for $|f|$:

$$\sqrt{\nu \epsilon} \leq \frac{\epsilon}{|u|} = \frac{(f, u)}{|u|} \leq \frac{|f||u|}{|u|} = |f|.$$  

• It is convenient to use this lower bound for $|f|$ to define a lower bound for the Grashof number $G = |f|/\nu^2$, which we use as the normalization $\tilde{G}$ for random forcing:

$$\tilde{G} = \sqrt{\frac{\epsilon}{\nu^3}}.$$  

• We recently proved the following theorem (JDE 2018):

**Theorem 2 (Emami & Bowman [2018])** For all $u \in A$ with energy injection rate $\epsilon$,

$$||u||^2 \leq \sqrt{\frac{\epsilon}{\nu}} |u|.$$  

• This leads to same form as for a constant force: $2Z \leq \nu \tilde{G} \sqrt{2E}$.
$Z - E$ Plane Bounds: Random Forcing

\[ \frac{2Z}{\nu^2G^2} \]

\[ \frac{2E}{\nu^2G^2} \]

A in here
DNS code

- We have released a highly optimized 2D pseudospectral code in C++: https://github.com/dealias/dns.

- It uses our FFTW++ library to implicitly dealias the advective convolution, while exploiting Hermitian symmetry [Bowman & Roberts 2011], [Roberts & Bowman 2018].

- Advanced computer memory management, such as implicit padding, memory alignment, and dynamic moment averaging allow DNS to attain its extreme performance.

- The formulation proposed by Basdevant [1983] is used to reduce the number of FFTs required for 2D (3D) incompressible turbulence to 4 (8).

- We also include simplified 2D (146 lines) and 3D (287 lines) versions called ProtoDNS for educational purposes: https://github.com/dealias/dns/tree/master/protodns.
Dynamic Moment Averaging

- Advantageous to precompute time-integrated moments like

\[ M_n(t) = \int_0^t |\omega_k(\tau)|^n \, d\tau. \]

- This can be accomplished done by evolving

\[ \frac{dM_n}{dt} = |\omega_k|^n, \]

along with the vorticity \( \omega_k \) itself, using \textit{the same} temporal discretization.

- These evolved quantities \( M_n \) can be used to extract accurate statistical averages during the post-processing phase, once the saturation time \( t_1 \) has been determined by the user:

\[ \int_{t_1}^{t_2} |\omega_k|^n(\tau) \, d\tau = M_n(t_2) - M_n(t_1). \]
Enstrophy Balance

\[ \frac{\partial \omega_k}{\partial t} + \nu k^2 \omega_k = S_k + f_k, \]

- Multiply by \( \omega_k^* \) and integrate over wavenumber angle \( \Rightarrow \)

  enstrophy spectrum \( Z(k) \) evolves as:

\[ \frac{\partial}{\partial t}Z(k) + 2\nu k^2 Z(k) = 2T(k) + G(k), \]

where \( T(k) \) and \( G(k) \) are the corresponding angular averages of \( \text{Re} \langle S_k \omega_k^* \rangle \) and \( \text{Re} \langle f_k \omega_k^* \rangle \).
Nonlinear Enstrophy Transfer Function

\[
\frac{\partial}{\partial t} Z(k) + 2\nu k^2 Z(k) = 2T(k) + G(k).
\]

- Let

\[
\Pi(k) \doteq 2 \int_k^\infty T(p) \, dp
\]

represent the nonlinear transfer of enstrophy into \([k, \infty)\).

- Integrate from \(k\) to \(\infty\):

\[
\frac{d}{dt} \int_k^\infty Z(p) \, dp = \Pi(k) - \epsilon_Z(k),
\]

where \(\epsilon_Z(k) \doteq 2\nu \int_k^\infty p^2 Z(p) \, dp - \int_k^\infty G(p) \, dp\) is the total enstrophy transfer, via dissipation and forcing, out of wavenumbers higher than \(k\).
• A positive (negative) value for $\Pi(k)$ represents a flow of enstrophy to wavenumbers higher (lower) than $k$.

• When $\nu = 0$ and $f_k = 0$:

$$0 = \frac{d}{dt} \int_0^\infty Z(p) \, dp = 2 \int_0^\infty T(p) \, dp,$$

so that

$$\Pi(k) = 2 \int_k^\infty T(p) \, dp = -2 \int_0^k T(p) \, dp.$$ 

• Note that $\Pi(0) = \Pi(\infty) = 0$.

• In a steady state, $\Pi(k) = \epsilon_Z(k)$.

• This provides an excellent numerical diagnostic for determining the saturation time $t_1$. 
Vorticity Field with Hypoviscosity
Energy Spectrum with Hypoviscosity

\[ E(k) \]

- \[ 10^{-6} \]
- \[ 10^{-8} \]
- \[ 10^{-10} \]
- \[ 10^{-12} \]

- \[ k \]
- \[ 10^0 \]
- \[ 10^1 \]
- \[ 10^2 \]
Bounds in the $Z-E$ Plane for Random Forcing

\begin{align*}
2Z/\langle \nu \tilde{G} \rangle^2 &= \frac{2}{\nu \tilde{G}^2} \quad (0.005, 0.01, 0.015) \\
2E/\langle \nu \tilde{G} \rangle^2 &= \frac{2E}{\nu \tilde{G}^2} \quad (0.1, 0.2, 0.3)
\end{align*}
Enstrophy Transfer with Hypoviscosity

Cumulative enstrophy transfer

$\Pi$  $\eta$

$k$

$10^0$  $10^1$  $10^2$
Vorticity Field without Hypoviscosity
Bounds in the $Z-E$ Plane for Random Forcing

![Graph showing bounds in the Z–E Plane for random forcing.](image-url)
Enstrophy Transfer without Hypoviscosity
Effect of Adding Friction

• Many numerical simulations of turbulence remove the energy from the large scales by adding a simple friction term $-\gamma u$:

$$\frac{\partial u}{\partial t} + \nu A u + B(u, u) = -\gamma u + f.$$ 

• Our analysis can be generalized to account for friction by redefining the effective Grashof number as

$$\tilde{G} = \sqrt{\epsilon (\nu + \gamma)}$$

which again leads to the upper bound

$$2Z \leq \nu \tilde{G} \sqrt{2E}.$$
Energy Spectrum with Friction

\[ E(k) \]

\[ \begin{array}{c}
10^{-13} \\
10^{-11} \\
10^{-9} \\
10^{-7} \\
10^{-5} \\
10^{-3} \\
10^{-1} \\
10^0 \\
10^1 \\
10^2 \\
\end{array} \]
Bounds in the $Z-E$ Plane with Friction

![Graph showing bounds in the $Z-E$ plane with friction.](image)
Special Case: White-Noise Forcing

- The Fourier transform of an isotropic Gaussian white-noise solenoidal force $\mathbf{f}$ has the form

\[ \mathbf{f}_k(t) = F_k \left( 1 - \frac{\mathbf{k} \cdot \mathbf{k}}{k^2} \right) \cdot \mathbf{\xi}_k(t), \quad \mathbf{k} \cdot \mathbf{f}_k = 0, \]

where $F_k$ is a real number and $\mathbf{\xi}_k(t)$ is a unit central real Gaussian random 2D vector that satisfies

\[ \langle \mathbf{\xi}_k(t) \mathbf{\xi}_k'(t') \rangle = \delta_{k,k'} \mathbf{1} \delta(t - t'). \]

- This implies

\[ \langle \mathbf{f}_k(t) \mathbf{f}_k'(t') \rangle = F_k^2 \delta_{k,k'} \delta(t - t'). \]
Special Case: White-Noise Forcing

- As in the constant forcing case, the rate of energy injection $\epsilon$ is given by

$$
\epsilon = (f(x, t), u(x, t)) = \int_{\Omega} \langle f(x, t) \cdot u(x, t) \rangle \, dx = \text{Re} \sum_k \langle f_k(t) \cdot \bar{u}_k(t) \rangle
$$

- Here $u_k(t)$ is functional of the forcing:

$$
u_k(t) = u_k(t') + \int_{t'}^{t} A_k[u(\tau)] \, d\tau + \int_{t'}^{t} f_k(\tau) \, d\tau,
$$

where $A_k$ is a functional of $u$ such that $\frac{\delta A_k[u(\tau)]}{\delta f_k'(t')}$ is bounded.

- Nonlinear Green’s function:

$$
\frac{\delta u_k(t)}{\delta f_k'(t')} = \int_{t'}^{t} \frac{\delta A_k[u(\tau)]}{\delta f_k'(t')} \, d\tau + \delta_{kk'} 1 H(t - t'),
$$

where $H$ is the Heaviside unit step function.
To prescribe the forcing amplitude $F_k$ in terms of $\epsilon$:

**Theorem 3 (Novikov [1964])** If $f(\mathbf{x}, t)$ is a Gaussian process, and $u$ is a functional of $f$, then

$$\langle f(\mathbf{x}, t)u(f) \rangle = \int \int \langle f(\mathbf{x}, t)f(\mathbf{x}', t') \rangle \left\langle \frac{\delta u(\mathbf{x}, t)}{\delta f(\mathbf{x}', t')} \right\rangle d\mathbf{x}' dt'.$$

For white-noise forcing:

$$\epsilon = \text{Re} \sum_k \langle f_k(t) \cdot \overline{u}_k(t) \rangle = \text{Re} \sum_{k,k'} \int \langle f_k(t) \overline{f}_k(t') \rangle : \left\langle \frac{\delta \overline{u}_k(t)}{\delta \overline{f}_k(t')} \right\rangle dt'.$$

$$= \sum_k F_k^2 \left( 1 - \frac{kk}{k^2} \right) : \left( 1 - \frac{kk}{k^2} \right) H(0)$$

$$= \frac{1}{2} \sum_k F_k^2,$$

on noting that $H(0) = 1/2$. 
White-Noise Forcing: Implementation

- At the end of each time-step, we implement the contribution of white noise forcing with the discretization

\[ \omega_{k,n+1} = \omega_{k,n} + \sqrt{2\tau \eta_k} \xi, \]

where \( \xi \) is a unit complex Gaussian random number with \( \langle \xi \rangle = 0 \) and \( \langle |\xi|^2 \rangle = 1 \).

- This yields the mean enstrophy injection

\[ \frac{\langle |\omega_{k,n+1}|^2 - |\omega_{k,n}|^2 \rangle}{2\tau} = \eta_k. \]
3D Basdevant Formulation: 8 FFTs

- Using incompressibility, the 3D momentum equation can be written in terms of the symmetric tensor $D_{ij} = u_i u_j$:

$$\frac{\partial u_i}{\partial t} + \frac{\partial D_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + F_i.$$ 

- Naive implementation: 3 backward FFTs to compute the velocity components from their spectral representations, 6 forward FFTs of the independent components of $D_{ij}$.

- Basdevant [1983]: avoid one FFT by subtracting the divergence of the symmetric matrix $S_{ij} = \delta_{ij} \text{tr} D/3$ from both sides:

$$\frac{\partial u_i}{\partial t} + \frac{\partial (D_{ij} - S_{ij})}{\partial x_j} = -\frac{\partial (p\delta_{ij} + S_{ij})}{\partial x_j} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + F_i.$$ 

- To compute the velocity components $u_i$, 3 backward FFTs are required. Since the symmetric matrix $D_{ij} - S_{ij}$ is traceless, it has just 5 independent components.
• Hence, a total of only 8 FFTs are required per integration stage.

• The effective pressure $p\delta_{ij} + S_{ij}$ is solved as usual from the inverse Laplacian of the force minus the nonlinearity.
2D Basdevant Formulation: 4 FFTs

- The vorticity $\omega = \nabla \times u$ evolves according to

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla) \omega = (\omega \cdot \nabla) u + \nu \nabla^2 \omega + \nabla \times F,$$

where in 2D the vortex stretching term $(\omega \cdot \nabla) u$ vanishes and $\omega$ is normal to the plane of motion.

- For $C^2$ velocity fields, the curl of the nonlinearity can be written in terms of $\tilde{D}_{ij} = D_{ij} - S_{ij}$:

$$\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_j} \tilde{D}_{2j} - \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_j} \tilde{D}_{1j} = \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) D_{12} + \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} (D_{22} - D_{11}),$$

on recalling that $S$ is diagonal and $S_{11} = S_{22}$.

- The scalar vorticity $\omega$ thus evolves as

$$\frac{\partial \omega}{\partial t} + \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) (u_1 u_2) + \frac{\partial^2}{\partial x_1 \partial x_2} (u_2^2 - u_1^2) = \nu \nabla^2 \omega + \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}.$$
To compute $u_1$ and $u_2$ in physical space, we need 2 backward FFTs.

The quantities $u_1 u_2$ and $u_2^2 - u_1^2$ can then be calculated and then transformed to Fourier space with 2 additional forward FFTs.

The advective term in 2D can thus be calculated with just 4 FFTs.
3D Incompressible MHD: 14 FFTs

\[
\frac{\partial u_i}{\partial t} + \frac{\partial (D_{ij} - S_{ij})}{\partial x_j} = -\frac{\partial (p\delta_{ij} + S_{ij})}{\partial x_j} + \nu \frac{\partial^2 u_i}{\partial x_j^2},
\]

\[
\frac{\partial B_i}{\partial t} + \frac{\partial G_{ij}}{\partial x_j} = \eta \frac{\partial^2 B_i}{\partial x_j^2},
\]

where \( D_{ij} = u_i u_j - B_i B_j, \ S_{ij} = \delta_{ij} \text{tr} \ D/3, \) and

\[ G_{ij} = B_i u_j - u_i B_j. \]

- The traceless symmetric matrix \( D_{ij} - S_{ij} \) has 5 independent components.
- The antisymmetric matrix \( G_{ij} \) has only 3.
- An additional 6 FFT calls are required to compute the components of \( \mathbf{u} \) and \( \mathbf{B} \) in \( x \) space.
- The MHD nonlinearity can thus be computed with 14 FFT calls.
Discrete Cyclic Convolution

• The FFT provides an efficient tool for computing the *discrete cyclic convolution*

\[ \sum_{p=0}^{N-1} F_p G_{k-p}, \]

where the vectors $F$ and $G$ have period $N$.

• The backward 1D *discrete Fourier transform* of a complex vector $\{F_k : k = 0, \ldots, N - 1\}$ is defined as

\[ f_j = \sum_{k=0}^{N-1} \zeta_N^{jk} F_k, \quad j = 0, \ldots, N - 1, \]

where $\zeta_N = e^{2\pi i/N}$ denotes the $N$th primitive root of unity.

• The fast Fourier transform (FFT) method exploits the properties that $\zeta_N^r = \zeta_{N/r}$ and $\zeta_N^N = 1$. 
Convolution Theorem

\[ \sum_{j=0}^{N-1} f_j g_j \zeta_N^{-jk} = \sum_{j=0}^{N-1} \zeta_N^{-jk} \left( \sum_{p=0}^{N-1} \zeta_N^{jp} F_p \right) \left( \sum_{q=0}^{N-1} \zeta_N^{jq} G_q \right) \]

\[ = \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} F_p G_q \sum_{j=0}^{N-1} \zeta_N^{(-k+p+q)j} \]

\[ = N \sum_{s} \sum_{p=0}^{N-1} F_p G_{k-p+sN}. \]

- The terms indexed by \( s \neq 0 \) are aliases; we need to remove them!

- If only the first \( m \) entries of the input vectors are nonzero, aliases can be avoided by zero padding input data vectors of length \( m \) to length \( N \geq 2m - 1 \).

- *Explicit zero padding* prevents mode \( m - 1 \) from beating with itself, wrapping around to contaminate mode \( N = 0 \mod N \).
Implicit Dealiasing

• Let $N = 2m$. For $j = 0, \ldots, 2m - 1$ we want to compute

$$f_j = \sum_{k=0}^{2m-1} \zeta_{2m}^{jk} F_k.$$

• If $F_k = 0$ for $k \geq m$, one can easily avoid looping over the unwanted zero Fourier modes by decimating in wavenumber:

$$f_{2\ell} = \sum_{k=0}^{m-1} \zeta_{2m}^{2\ell k} F_k = \sum_{k=0}^{m-1} \zeta_{m}^{\ell k} F_k,$$

$$f_{2\ell+1} = \sum_{k=0}^{m-1} \zeta_{2m}^{(2\ell+1)k} F_k = \sum_{k=0}^{m-1} \zeta_{m}^{\ell k} \zeta_{2m}^{k} F_k, \quad \ell = 0, 1, \ldots, m - 1.$$

• This requires computing two subtransforms, each of size $m$, for an overall computational scaling of order $2m \log_2 m = N \log_2 m$. 

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Parallelized multidimensional implicit dealiasing routines have been implemented as a software layer **FFTW++** (v2.05) on top of the **FFTW** library under the Lesser GNU Public License:

http://fftwpp.sourceforge.net/

\[
\{F_k\}_{k=0}^{m-1} \quad \{G_k\}_{k=0}^{m-1}
\]
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Conclusions

• The upper bound in the $Z-E$ plane obtained previously for constant forcing also works for white-noise forcing.

• Adding a large-scale hypoviscosity to the Navier–Stokes equation has a dramatic effect on the turbulent dynamics: it restricts the global attractor to the region characterized by the forcing annulus.

• The bounds on the attractor can easily be generalized to handle a friction term acting on all scales (instead of a large-scale hypoviscosity).

• With added friction, the observed dynamics lies well within the bounds on the attractor.

• We plan to study the relation between white-noise and constant forcings by examining their effects on the global attractor.

• Such analytical bounds provide a means to evaluate various heuristic turbulent subgrid (and supergrid!) models.
References


