On the Global Attractor of 2D Incompressible Turbulence with Random Forcing

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August 19, 2017

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Turbulence

Big whirls have little whirls that feed on their velocity, and little whirls have littler whirls and so on to viscosity... [Richardson 1922]

- In 1941, Kolmogorov conjectured that the energy spectrum of 3D incompressible turbulence exhibits a self-similar power-law scaling characterized by a uniform cascade of energy to molecular (viscous) scales:

\[ E(k) = C \epsilon^{2/3} k^{-5/3}. \]

- Here \( k \) is the Fourier wavenumber and \( E(k) \) is normalized so that \( \int E(k) \, dk \) is the total energy.

- Kolmogorov suggested that \( C \) might be a universal constant.
3D Energy Cascade

\[ \log E(k) \quad \log k \]

Forcing Range
Inertial Range
Dissipation Range

\[ k^{-5/3} \]

\[ k_f \]

\[ k_d \]
2D Incompressible Turbulence

• In 2D, where $\mathbf{u}$ maps a plane normal to $\mathbf{\hat{z}}$ to $\mathbb{R}^2$, the vorticity vector $\mathbf{\omega} = \nabla \times \mathbf{u}$ is always perpendicular to $\mathbf{u}$.

• Navier–Stokes equation for the scalar vorticity $\omega = \mathbf{\hat{z}} \cdot \nabla \times \mathbf{u}$:

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = -\nu \nabla^2 \omega + f.$$ 

• The incompressibility condition $\nabla \cdot \mathbf{u} = 0$ can be exploited to find $\mathbf{u}$ in terms of $\omega$:

$$\nabla \omega \times \mathbf{\hat{z}} = \nabla \times \mathbf{\hat{z}} \omega = \nabla \times (\nabla \times \mathbf{u}) = \nabla (\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u} = -\nabla^2 \mathbf{u}.$$ 

• Thus $\mathbf{u} = \mathbf{\hat{z}} \times \nabla \nabla^{-2} \omega$. In Fourier space:

$$\frac{d\omega_k}{dt} = S_k - \nu k^2 \omega_k + f_k,$$

where $S_k = \sum_q \frac{\mathbf{\hat{z}} \times q \cdot k}{q^2} \bar{\omega}_q \bar{\omega}_{-k-q} = \sum_{p,q} \epsilon_{kqp} \frac{\omega_p \omega_q}{q^2 \omega_q}$.
Here $\epsilon_{kpq} = \hat{z} \cdot \mathbf{p} \times \mathbf{q} \, \delta_{k+p+q}$ is antisymmetric under permutation of any two indices.

$$\frac{d\omega_k}{dt} + \nu k^2 \omega_k = \sum_p \sum_q \frac{\epsilon_{kpq}}{q^2} \omega_p \omega_q + f_k,$$

- When $\nu = f_k = 0$:

enstrophy $Z = \frac{1}{2} \sum_k |\omega_k|^2$ and energy $E = \frac{1}{2} \sum_k \frac{|\omega_k|^2}{k^2}$ are conserved:

$$\frac{\epsilon_{kpq}}{q^2} \quad \text{antisymmetric in} \quad k \leftrightarrow p,$$

$$\frac{1}{k^2} \frac{\epsilon_{kpq}}{q^2} \quad \text{antisymmetric in} \quad k \leftrightarrow q.$$
Fjørtoft Dual Cascade Scenario

\[ E_2 = E_1 + E_3, \quad Z_2 = Z_1 + Z_3, \quad Z_i \approx k_i^2 E_i. \]

- When \( k_1 = k, k_2 = 2k, \) and \( k_3 = 4k:\)
  \[ E_1 \approx \frac{4}{5} E_2, \quad Z_1 \approx \frac{1}{5} Z_2, \quad E_3 \approx \frac{1}{5} E_2, \quad Z_3 \approx \frac{4}{5} Z_2. \]

- Fjørtoft [1953]: energy cascades to large scales and enstrophy cascades to small scales.
2D Energy Cascade

\[ \log E(k) \]

- Forcing Range: \( k^{-5/3} \)
- Inertial Range: \( k^{-3} \)
- Dissipation Range

\( k_f \) and \( k_d \) are the limits of the ranges.
2D Turbulence: Mathematical Formulation

- Consider the Navier–Stokes equations for 2D incompressible homogeneous isotropic turbulence with density $\rho = 1$:

$$
\frac{\partial u}{\partial t} - \nu \nabla^2 u + u \cdot \nabla u + \nabla P = F, \\
\nabla \cdot u = 0, \\
\int_{\Omega} u \, dx = 0, \quad \int_{\Omega} F \, dx = 0, \\
u(x, 0) = u_0(x),
$$

with $\Omega = [0, 2\pi] \times [0, 2\pi]$ and periodic boundary conditions on $\partial \Omega$.

- Introduce the Hilbert space

$$
H(\Omega) \doteq \text{cl} \left\{ u \in (C^2(\Omega) \cap L^2(\Omega))^2 \mid \nabla \cdot u = 0, \int_{\Omega} u \, dx = 0 \right\}.
$$

with inner product $(u, v) = \int_{\Omega} u(x, t) \cdot v(x, t) \, dx$ and $L^2$ norm $|u| = (u, u)^{1/2}$.
• For \( \mathbf{u} \in H(\Omega) \), the Navier–Stokes equations can be expressed:

\[
\frac{d\mathbf{u}}{dt} - \nu \nabla^2 \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \mathbf{F}.
\]

• Introduce \( A \doteq -\mathcal{P}(\nabla^2) \), \( f \doteq \mathcal{P}(\mathbf{F}) \), and the bilinear map

\[
\mathcal{B}(\mathbf{u}, \mathbf{u}) \doteq \mathcal{P}(\mathbf{u} \cdot \nabla \mathbf{u} + \nabla P),
\]

where \( \mathcal{P} \) is the Helmholtz–Leray projection operator from \((L^2(\Omega))^2\) to \(H(\Omega)\):

\[
\mathcal{P}(\mathbf{v}) \doteq \mathbf{v} - \nabla \nabla^{-2} \nabla \cdot \mathbf{v}, \quad \forall \mathbf{v} \in (L^2(\Omega))^2.
\]

• The dynamical system can then be compactly written:

\[
\frac{d\mathbf{u}}{dt} + \nu A \mathbf{u} + \mathcal{B}(\mathbf{u}, \mathbf{u}) = \mathbf{f}.
\]
Stokes Operator $A$

- The operator $A = \mathcal{P}(-\nabla^2)$ is positive semi-definite and self-adjoint, with a compact inverse.

- On the periodic domain $\Omega = [0, 2\pi] \times [0, 2\pi]$, the eigenvalues of $A$ are

$$\lambda = k \cdot k, \quad k \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\}.$$

- The eigenvalues of $A$ can be arranged as

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \quad \lambda_0 = 1$$

and its eigenvectors $\mathbf{w}_i, \ i \in \mathbb{N}_0$, form an orthonormal basis for the Hilbert space $H$, upon which we can define any quotient power of $A$:

$$A^\alpha \mathbf{w}_j = \lambda_j^\alpha \mathbf{w}_j, \quad \alpha \in \mathbb{R}, \quad j \in \mathbb{N}_0.$$
Subspace of Finite Enstrophy

- We define the subspace of $H$ consisting of solutions with finite enstrophy:

$$V = \left\{ u \in H \mid \sum_{j=0}^{\infty} \lambda_j(u, w_j)^2 < \infty \right\}.$$

- Another suitable norm for elements $u \in V$ is

$$\|u\| = \left| A^{1/2}u \right| = \left( \int_{\Omega} \sum_{i=1}^{2} \frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_i} \right)^{1/2} = \left( \sum_{j=0}^{\infty} \lambda_j(u, w_j)^2 \right)^{1/2}.$$
Properties of the Bilinear Map

• We will make use of the antisymmetry

\[(\mathcal{B}(u, v), w) = -(\mathcal{B}(u, w), v).\]

• In 2D, we also have orthogonality:

\[(\mathcal{B}(u, u), Au) = 0\]

and the strong form of enstrophy invariance:

\[(\mathcal{B}(Av, v), u) = (\mathcal{B}(u, v), Av).\]

• In 2D the above properties imply the symmetry

\[(\mathcal{B}(Au, u), u) + (\mathcal{B}(v, Av), u) + (\mathcal{B}(v, v), Av) = 0.\]

• We will need the 2D Ladyzhenskaya inequality

\[|u|_{L^4(\Omega)} \leq C_L |u|^{1/2}||u||^{1/2},\]

where the constant \(C_L\) depends only on the domain \(\Omega\).
Dynamical Behaviour

- Our starting point is the incompressible 2D Navier–Stokes equation with periodic boundary conditions:

\[
\frac{du}{dt} + \nu Au + B(u, u) = f, \quad u \in H.
\]

- Take the inner product with \( u \) (respectively \( Au \)):

\[
\frac{1}{2} \frac{d}{dt} |u(t)|^2 + \nu \|u(t)\|^2 = (f, u(t)),
\]

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + \nu \|Au(t)\|^2 = (f, Au(t)).
\]

- The Cauchy–Schwarz and Poincaré inequalities yield

\[(f, u(t)) \leq |f| \|u(t)\| \quad \text{and} \quad |u(t)| \leq \|u(t)\|.
\]

- Since the existence and uniqueness for solutions to the 2D Navier–Stokes equation has been proven, a global attractor can be defined ? , ?.
Dynamical Behaviour: Constant Forcing

• If the force $f$ is constant with respect to time, a Gronwall inequality can be exploited:

$$|u(t)|^2 \leq e^{-\nu t}|u(0)|^2 + (1 - e^{-\nu t}) \left(\frac{|f|}{\nu}\right)^2.$$ 

• Defining a nondimensional Grashof number $G = \frac{|f|}{\nu^2}$, the above inequality can be simplified to

$$|u(t)|^2 \leq e^{-\nu t}|u(0)|^2 + (1 - e^{-\nu t})\nu^2 G^2.$$ 

• Similarly,

$$||u(t)||^2 \leq e^{-\nu t}||u(0)||^2 + (1 - e^{-\nu t})\nu^2 G^2.$$ 

• Being on the attractor thus requires

$$|u| \leq \nu G \quad \text{and} \quad ||u|| \leq \nu G.$$
Attractor Set $\mathcal{A}$

• Let $S$ be the solution operator:

$$S(t)u_0 = u(t), \quad u_0 = u(0),$$

where $u(t)$ is the unique solution of the Navier–Stokes equations.

• The closed ball $\mathcal{B}$ of radius $\nu G$ in the space $V$ is a bounded absorbing set in $H$.

• That is, for any bounded set $\mathcal{B}'$ there exists a time $t_0$ such that

$$t_0 = t_0(\mathcal{B}'), \quad \text{and} \quad S(t)\mathcal{B}' \subset \mathcal{B}, \quad \forall t \geq t_0.$$

• We can then construct the global attractor:

$$\mathcal{A} = \bigcap_{t \geq 0} S(t)\mathcal{B},$$

so $\mathcal{A}$ is the largest bounded, invariant set such that $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$. 

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This text is a concise summary of the concepts related to attractor sets in the context of the Navier–Stokes equations. The attractor set $\mathcal{A}$ is defined as the intersection of all solution operators $S(t)\mathcal{B}$, where $\mathcal{B}$ is a bounded absorbing set in the space $H$. The existence of such an attractor is a fundamental result in the study of dynamical systems and partial differential equations.
**Z–E Plane Bounds: Constant Forcing**

- A trivial lower bound is provided by the Poincaré inequality:

\[ |u|^2 \leq \|u\|^2 \Rightarrow E \leq Z. \]

- An upper bound is given by

**Theorem 1 (Dascaliuc, Foias, and Jolly [2005])**

*For all* \( u \in A \),

\[ \|u\|^2 \leq \frac{|f|}{\nu} |u|. \]

- That is,

\[ Z \leq \nu G \sqrt{E}. \]
$Z - E$ Plane Bounds: Constant Forcing

\[ \frac{2Z}{\nu^2 G^2} \quad \text{and} \quad \frac{2E}{\nu^2 G^2} \]

A in here
Extended Norm: Random Forcing

- For a random variable $\alpha$, with probability density function $P$, define the ensemble average

$$\langle \alpha \rangle = \int_{-\infty}^{\infty} \alpha \left( \frac{dP}{d\zeta} \right) d\zeta.$$  

- The extended inner product is

$$(u, v)_{\tilde{\omega}} = \int_\Omega \langle u \cdot v \rangle \, dx = \int_\Omega \left( \int_{-\infty}^{\infty} u \cdot v \frac{dP}{d\zeta} d\zeta \right) dx,$$

with norm

$$|f|_{\tilde{\omega}} \doteq \left( \int_\Omega \langle |f|^2 \rangle \, dx \right)^{1/2}.$$
Dynamical Behaviour: Random Forcing

- Energy balance:

\[ \frac{1}{2} \frac{d}{dt} |u|^2 + \nu (Au, u) + (B(u, u), u) = (f, u) - \epsilon, \]

where \( \epsilon \) is the rate of energy injection.

- From the energy conservation identity \( (B(u, u), u) = 0 \),

\[ \frac{1}{2} \frac{d}{dt} |u|^2 + \nu ||u||^2 = \epsilon. \]

- The Poincaré inequality \( ||u|| \geq |u| \) leads to

\[ \frac{1}{2} \frac{d}{dt} |u|^2 \leq \epsilon - \nu |u|^2, \]

which implies that \( |u(t)|^2 \leq e^{-2\nu t} |u(0)|^2 + \left( \frac{1 - e^{-2\nu t}}{\nu} \right) \epsilon. \)

- So for every \( u \in A \), we expect \( |u(t)|^2 \leq \epsilon/\nu. \)
• From $|\mathbf{u}(t)| \leq \sqrt{\epsilon}/\nu$ we then obtain a lower bound for $|f|:
\sqrt{\nu\epsilon} \leq \frac{\epsilon}{|\mathbf{u}|} = \frac{(f, \mathbf{u})}{|\mathbf{u}|} \leq \frac{|f||\mathbf{u}|}{|\mathbf{u}|} = |f|.

• It is convenient to use this lower bound for $|f|$ to define a lower bound for the Grashof number $G = |f|/\nu^2$, which we use as the normalization $\tilde{G}$ for random forcing:
$$
\tilde{G} = \sqrt{\frac{\epsilon}{\nu^3}}.
$$

• We recently proved the following theorem (submitted to JDE):

**Theorem 2 (?)** For all $\mathbf{u} \in A$ with energy injection rate $\epsilon$,
$$
||\mathbf{u}||^2 \leq \sqrt{\frac{\epsilon}{\nu}}|\mathbf{u}|.
$$

• This leads to the same form as for a constant force: $Z \leq \nu\tilde{G}\sqrt{E}$.
$Z - E$ Plane Bounds: Random Forcing

\[
\frac{2Z}{\nu^2 G^2} \quad \text{and} \quad 1 \quad \frac{2E}{\nu^2 G^2}
\]

A in here
DNS code

• We have released a highly optimized 2D pseudospectral code in C++: https://github.com/dealias/dns.

• It uses our FFTW++ library to implicitly dealias the advective convolution, while exploiting Hermitian symmetry ? , ?.

• Advanced computer memory management, such as implicit padding, memory alignment, and dynamic moment averaging allow DNS to attain its extreme performance.

• It uses the formulation proposed by ? to reduce the number of FFTs required for 2D (3D) incompressible turbulence to 4 (8).

• We also include simplified 2D (146 lines) and 3D (287 lines) versions called ProtoDNS for educational purposes: https://github.com/dealias/dns/tree/master/protodns.
Vorticity Field with Hypoviscosity
Energy Spectrum with Hypoviscosity
Bounds in the $Z-E$ plane for random forcing.
Energy Transfer with Hypoviscosity

Cumulative enstrophy transfer

$k$

$\Pi$

$\eta$
Vorticity Field without Hypoviscosity
Energy Spectrum without Hypoviscosity

\[ E(k) \]

\[ k \]

\[ 10^{-11} \quad 10^{-9} \quad 10^{-7} \quad 10^{-5} \quad 10^{-3} \quad 10^{-1} \quad 10^0 \quad 10^1 \quad 10^2 \]
Bounds in the $Z-E$ plane for random forcing.
Energy Transfer without Hypoviscosity

Cumulative enstropy transfer vs. $k$

- Blue line: $\Pi$
- Red line: $\eta$

Cumulative enstropy transfer

$k$

$10^0$, $10^1$, $10^2$
Special Case: White-Noise Forcing

- The Fourier transform of an isotropic Gaussian white-noise solenoidal force $f$ has the form

$$f_k(t) = F_k \left(1 - \frac{k k}{k^2}\right) \cdot \xi_k(t), \quad k \cdot f_k = 0,$$

where $F_k$ is a real number and $\xi_k(t)$ is a unit central real Gaussian random 2D vector that satisfies

$$\langle \xi_k(t) \xi_{k'}(t') \rangle = \delta_{k,k'} 1 \delta(t - t').$$

- This implies

$$\langle f_k(t) \cdot f_{k'}(t') \rangle = F_k^2 \delta_{k,k'} \delta(t - t').$$
Special Case: White-Noise Forcing

• As in the constant forcing case, the rate of energy injection $\epsilon$ is given by

$$\epsilon = \langle f(x,t), u(x,t) \rangle = \int_\Omega \langle f(x,t) \cdot u(x,t) \rangle \, dx = \text{Re} \sum_k \langle f_k(t) \cdot \overline{u}_k(t) \rangle$$

• Here $u_k(t)$ is functional of the forcing:

$$u_k(t) = u_{k'}(t') + \int_{t'}^t A_k[u(\tau)] \, d\tau + \int_{t'}^t f_k(\tau) \, d\tau,$$

where $A_k$ is a functional of $u$ such that $\frac{\delta A_k[u(\tau)]}{\delta f_{k'}(t')}$ is bounded.

• Nonlinear Green’s function:

$$\frac{\delta u_k(t)}{\delta f_{k'}(t')} = \int_{t'}^t \frac{\delta A_k[u(\tau)]}{\delta f_{k'}(t')} \, d\tau + \delta_{kk'} 1 H(t - t'),$$

where $H$ is the Heaviside unit step function.
To prescribe the forcing amplitude $F_k$ in terms of $\epsilon$:

**Theorem 3 (?)** If $f(x, t)$ is a Gaussian process, and $u$ is a functional of $f$, then

$$
\langle f(x, t)u(f) \rangle = \int \int \langle f(x, t)f(x', t') \rangle \left\langle \frac{\delta u(x, t)}{\delta f(x', t')} \right\rangle \, dx' \, dt'.
$$

For white-noise forcing, we obtain

\[
\epsilon = \text{Re} \sum_k \langle f_k(t) \cdot \overline{u}_k(t) \rangle = \text{Re} \sum_{k,k'} \int \langle f_k(t) \overline{f}_{k'}(t') \rangle : \left\langle \frac{\delta \overline{u}_k(t)}{\delta \overline{f}_{k'}(t')} \right\rangle \, dt' \\
= \sum_k F_k^2 \left( 1 - \frac{kk}{k^2} \right) : \left( 1 - \frac{kk}{k^2} \right) H(0) \\
= \frac{1}{2} \sum_k F_k^2,
\]

on noting that $H(0) = 1/2$. 
3D Basdevant Formulation: 8 FFTs

- Using incompressibility, the 3D momentum equation can be written in terms of the symmetric tensor $D_{ij} = u_i u_j$:

$$\frac{\partial u_i}{\partial t} + \frac{\partial D_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + F_i.$$ 

- Naive implementation: 3 backward FFTs to compute the velocity components from their spectral representations, 6 forward FFTs of the independent components of $D_{ij}$.

- $?$: avoid one FFT by subtracting the divergence of the symmetric matrix $S_{ij} = \delta_{ij} \text{tr} D/3$ from both sides:

$$\frac{\partial u_i}{\partial t} + \frac{\partial (D_{ij} - S_{ij})}{\partial x_j} = -\frac{\partial (p\delta_{ij} + S_{ij})}{\partial x_j} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + F_i.$$ 

- To compute the velocity components $u_i$, 3 backward FFTs are required. Since the symmetric matrix $D_{ij} - S_{ij}$ is traceless, it has just 5 independent components.
• Hence, a total of only 8 FFTs are required per integration stage.

• The effective pressure $p\delta_{ij} + S_{ij}$ is solved as usual from the inverse Laplacian of the force minus the nonlinearity.
2D Basdevant Formulation: 4 FFTs

- The vorticity \( \omega = \nabla \times \mathbf{u} \) evolves according to

\[
\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla) \omega = (\omega \cdot \nabla) \mathbf{u} + \nu \nabla^2 \omega + \nabla \times \mathbf{F},
\]

where in 2D the vortex stretching term \((\omega \cdot \nabla) \mathbf{u}\) vanishes and \(\omega\) is normal to the plane of motion.

- For \(C^2\) velocity fields, the curl of the nonlinearity can be written in terms of \(\tilde{D}_{ij} = D_{ij} - S_{ij}\):

\[
\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_j} \tilde{D}_{2j} - \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_j} \tilde{D}_{1j} = \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) D_{12} + \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} (D_{22} - D_{11}),
\]

on recalling that \(S\) is diagonal and \(S_{11} = S_{22}\).

- The scalar vorticity \(\omega\) thus evolves as

\[
\frac{\partial \omega}{\partial t} + \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) (u_1u_2) + \frac{\partial^2}{\partial x_1 \partial x_2} (u_2^2 - u_1^2) = \nu \nabla^2 \omega + \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}.
\]
To compute $u_1$ and $u_2$ in physical space, we need 2 backward FFTs.

The quantities $u_1 u_2$ and $u_2^2 - u_1^2$ can then be calculated and then transformed to Fourier space with 2 additional forward FFTs.

The advective term in 2D can thus be calculated with just 4 FFTs.
Discrete Cyclic Convolution

- The FFT provides an efficient tool for computing the discrete cyclic convolution
  \[
  \sum_{p=0}^{N-1} F_p G_{k-p},
  \]
  where the vectors \( F \) and \( G \) have period \( N \).

- The backward 1D discrete Fourier transform of a complex vector \( \{F_k : k = 0, \ldots, N - 1\} \) is defined as
  \[
  f_j = \sum_{k=0}^{N-1} \zeta_N^{-jk} F_k, \quad j = 0, \ldots, N - 1,
  \]
  where \( \zeta_N = e^{2\pi i/N} \) denotes the \( N \)th primitive root of unity.

- The fast Fourier transform (FFT) method exploits the properties that \( \zeta_N^r = \zeta_N^{N/r} \) and \( \zeta_N^N = 1 \).
**Convolution Theorem**

\[
\sum_{j=0}^{N-1} f_j g_j \zeta_N^{-jk} = \sum_{j=0}^{N-1} \zeta_N^{-jk} \left( \sum_{p=0}^{N-1} \zeta_N^{jp} F_p \right) \left( \sum_{q=0}^{N-1} \zeta_N^{jq} G_q \right)
\]

\[
= \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} F_p G_q \sum_{j=0}^{N-1} \zeta_N^{(-k+p+q)j}
\]

\[
= N \sum_{s} \sum_{p=0}^{N-1} F_p G_{k-p+sN}.
\]

- The terms indexed by \( s \neq 0 \) are *aliases*; we need to remove them!

- If only the first \( m \) entries of the input vectors are nonzero, aliases can be avoided by *zero padding* input data vectors of length \( m \) to length \( N \geq 2m - 1 \).

- *Explicit zero padding* prevents mode \( m - 1 \) from beating with itself, wrapping around to contaminate mode \( N = 0 \mod N \).
Implicit Dealiasing

• Let $N = 2m$. For $j = 0, \ldots, 2m - 1$ we want to compute

$$f_j = \sum_{k=0}^{2m-1} \zeta_{2m}^j F_k.$$ 

• If $F_k = 0$ for $k \geq m$, one can easily avoid looping over the unwanted zero Fourier modes by decimating in wavenumber:

$$f_{2\ell} = \sum_{k=0}^{m-1} \zeta_{2m}^{2\ell k} F_k = \sum_{k=0}^{m-1} \zeta_{m}^{\ell k} F_k,$$

$$f_{2\ell+1} = \sum_{k=0}^{m-1} \zeta_{2m}^{(2\ell+1)k} F_k = \sum_{k=0}^{m-1} \zeta_{m}^{\ell k} \zeta_{2m}^{k} F_k, \quad \ell = 0, 1, \ldots m - 1.$$ 

• This requires computing two subtransforms, each of size $m$, for an overall computational scaling of order $2m \log_2 m = N \log_2 m$. 
Parallelized multidimensional implicit dealiasing routines have been implemented as a software layer FFTW++ (v 2.05) on top of the FFTW library under the Lesser GNU Public License:

http://fftwpp.sourceforge.net/

\[
\{F_k\}_{k=0}^{m-1} \quad \{G_k\}_{k=0}^{m-1}
\]
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\[
\begin{align*}
\{F_k\}_{k=0}^{m-1} & \quad \{G_k\}_{k=0}^{m-1} \\
\{f_{2\ell}\}_{\ell=0}^{m-1} & \quad \{f_{2\ell+1}\}_{\ell=0}^{m-1} \\
\{f_{2\ell}g_{2\ell}\}_{\ell=0}^{m-1} & \quad \{f_{2\ell+1}g_{2\ell+1}\}_{\ell=0}^{m-1} \\
\{(F \star G)_k\}_{k=0}^{m-1}
\end{align*}
\]
Conclusions

- The upper bound in the $Z-E$ plane obtained for constant forcing also works for the white-noise forcing.
- Adding hypoviscosity to the Navier–Stokes equation has a dramatic effect on the turbulent dynamics: it restricts the global attractor to the region characterized by the forcing annulus.
- With these tools, it should now be possible to study the relation between white-noise and constant forcings by examining their effects on the global attractor.
- This may lead to an explicit relation for the energy and enstrophy injection rates for constant forcing.
- Analytical bounds for random forcing provide a means to evaluate various heuristic turbulent subgrid (and supergrid!) models by characterizing the behaviour of the global attractor under these models.