MODELLING SEDIMENT DEPOSITION PATTERNS ARISING FROM SUDDENLY RELEASED FIXED-VOLUME TURBULENT SUSPENSIONS

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Abstract

Models presented in several recent papers [1–3] dealing with particle transport by, and deposition from, bottom gravity currents produced by the sudden release of dilute, well-mixed fixed-volume suspensions have been relatively successful in duplicating the experimentally observed long-time, distal, areal density of the deposit on a rigid horizontal bottom. These models, however, fail in their ability to capture the experimentally observed proximal pattern of the areal density with its pronounced dip in the region initially occupied by the well-mixed suspension and its equally pronounced local maximum at roughly the one third point of the total reach of the deposit. The central feature of the models employed in [1–3] is that the particles are always assumed to be vertically well-mixed by fluid turbulence and to settle out through the bottom viscous sublayer with the Stokes settling velocity for a fluid at rest with no re-entrainment of particles from the floor of the tank. Since this process is assumed from the outset in the models of [1–3], the numerical simulations for a fixed-volume release will not take into account the actual experimental conditions that prevail at the time of release of a well-mixed fixed-volume suspension. That is, owing to the vigorous stirring that produces the well-mixed suspension, the release volume will initially possess greater turbulent energy than does an unstirred release volume, which may only acquire turbulent energy as a result of its motion after release through various instability mechanisms. The eddy motion in the imposed fluid turbulence reduces the particle settling rates from the values that would be observed in an unstirred release volume possessing zero initial turbulent energy.

We here develop a model for particle bearing gravity flows initiated by the sudden release of a fixed-volume suspension that takes into account the initial turbulent energy of mixing in the release volume by means of a modified settling velocity that, over a time scale characteristic of turbulent energy decay, approaches the full Stokes settling velocity. Thereafter in the flow regime we assume that the turbulence persists and, in accord with current understanding concerning the mechanics of dense underflows, that this turbulence is most intense in the wall region at the bottom of the flow and relatively coarse and on the verge of collapse [22] at the top of the flow where the density contrast is compositionally maintained. We capture this behaviour by specifying a 'shape function' that is based upon experimental observations and provides for vertical structure in the volume fraction of particles present in the flow. The assumption of vertically well-mixed particle suspensions employed in [1-5] corresponds to a constant shape function equal to unity.

Combining these two refinements concerning the settling velocity and vertical structure of the volume fraction of particles into the conservation law for particles and coupling this with the fluid equations for a two-layer system we find that our results for areal density of deposits from sudden releases of fixed-volume suspensions are in excellent qualitative agreement with the experimentally determined areal densities of deposit as reported in [1,3,6]. In particular, our model does what none of the other models do in that it captures and explains the proximal depression in the areal density of deposit.

1. Introduction

A gravity current is the flow of one fluid within another when this flow takes place because of relatively small differences in density between the fluids [7]. In natural settings such as lakes, oceans or even reservoirs, there are a myriad of possible contributors to these density differences including temperature differences, salinity contrasts, suspended material, both organic and inorganic, as well as combinations of these mechanisms.

Gravity currents are primarily horizontal, occurring as either top or bottom boundary currents or as intrusions at some intermediate level. Turbidity currents are gravity currents in which the excess density or unit weight providing the driving buoyancy force is due to the presence of sediment being held in suspension by fluid turbulence. Thus whether or not a sediment-bearing gravity current is a turbidity current depends to a great extent upon the ordering that exists between the density of the interstitial (suspending) fluid and that of the ambient fluid [4]. For example, in the case of a bottom particle-bearing gravity current with an interstitial fluid of density less than or equal to that of the ambient, the driving buoyancy forces are due solely to the presence of suspended particles and we have a turbidity current. In this case it is possible that particle settling may lead to a reduction in bulk density of the bottom flow, thereby transforming the bottom current into an intrusive turbidity current at some neutrally buoyant intermediate depth. Further particle settling can subsequently lead to complete buoyancy reversal, the initiation of a buoyant plume [5], and the creation of a surface gravity current in which the remaining particles play no role in the dynamics. On the other hand, should the density of the interstitial fluid in the bottom current be greater than that of the ambient, then the turbidity and/or particle-bearing gravity current will remain buoyantly stable throughout the particle-settling phase. If this suspension either is initially or becomes sufficiently dilute through the settling of particles, then there will be a stage when these particles will play little or no role in the dynamics of the flow. The flow dynamics of such a particle-bearing, compositionally driven gravity current [4] will be governed by the density difference that exists between interstitial and ambient fluids.

Moodie *et al.* [4] noted an important difference between the case in which the densities of interstitial and ambient fluids are the same, so that only the suspended particles provide the driving buoyancy forces and the case in which the density of the interstitial fluid exceeds that of the ambient, so that a density contrast is maintained throughout the particle settling phase. In the latter case a consistent shallow-water model can be developed because the buoyancy of the current may be scaled relative to the buoyancy difference between the two fluids and this provides a constant reference value. As particles sediment from the lower layer, the buoyancy contrast tends to this constant reference value, thereby enabling the precepts of shallow-water theory to apply throughout the region in which sedimentation occurs. In contrast, a consistent shallow-water theory cannot be achieved in the former case in which the particles provide the sole buoyancy contrast. For in this case the buoyancy will be scaled with the initial density difference between the particle-bearing current and the surrounding homogeneous fluid. This scaling for the velocity field, when combined with the low-aspect ratio of the flow, demands that the majority of the particles settle out over a nondimensional distance $O(1)$. This means that the buoyancy contrast between the turbidity current and the upper layer will be lost over the same horizontal distance, resulting in a slowing and deepening of the current, thereby producing significant nonhydrostatic vertical pressure gradients. We shall return to this issue later in our introduction when we review some of the recent papers on the subject.

These turbidity and/or particle-bearing gravity currents with their potentially extremely complex dynamics occur in a vast array of natural and human-made settings. A relatively complete catalogue of events in which they play a role is to be found in the excellent account by Simpson [8]. They are of particular interest to oceanographers and geologists in that the abyssal plains of many oceans consist of sand and silt layers that appear to have come from the continental shelf, transported in the form of turbidity currents. Many sandstones previously believed to have been deposited in shallow water over long time periods were in fact deposited rather precipitously by turbidity currents in water thousands of meters deep [7]. These turbidity currents may be initiated when an unstable submarine shelf near a coastline collapses resulting in a submarine landslide [3,9]. This landslide entrains fluid as it moves downslope and this fluid suspends sediment and, when it reaches the basal plain, spreads as a gravity current depositing its sediment [3,10]. The laboratory experiments to be modelled here, in which a fixed volume of fluid containing suspended particles and possessing turbulent energy is released into an ambient fluid, is a very reasonable model for the situation described above, wherein a turbulent mass of particle-laden fluid debauches onto the abyssal plain of some ocean.

This paper, which represents an extension of our previous work [4], is devoted to a closer study of aspects of particle-bearing gravity currents in order to bring the theory into closer agreement with the recently reported experimental observations of Bonnecaze *et al.* [1,3] and Gladstone *et al.* [6]. It is then in turn felt that, in as much as these experiments are designed to elucidate the behaviour seen in natural phenomena, we shall be able to provide some improved insights into possible mechanisms operative in nature, such as those that lead to the observed properties of some turbidites. Although we have outlined the history of gravity current research in our previous paper [4] we feel that it is necessary for our interpretation and extension of previous results to recap some of those results here, particularly those results that have established certain paradigms in the subject.

The majority of the theoretical work on gravity currents from the early calculation by von Kármán [11] up to that by Benjamin [12] and right on through into the mid 1980s treated the gravity current as steady, existing in either an inertia-buoyancy or, at later stages in the flow, in a viscous-buoyancy balance or, in the very late stages of the flow when the current has become very thin, in a viscous-surface tension balance [13]. Exceptions to this approach have been discussed in [4].

In the realm of time-dependent homogeneous gravity flow studies that relate directly to our current analysis is the work of Rottman and Simpson [14] on instantaneous releases of bottom gravity flows. In a series of detailed experiments they studied instantaneous releases for $0 < h_0/H \leq 1$, where h_0 is the initial depth of the released fixed volume of heavy fluid and H the total depth of the fluid in the rectangular channel. In their work they concentrated on the transition of the flow to the self-similar phase. The key feature of their observations was that for h_0/H equal to or slightly less than unity, the disturbance generated at the end wall has the appearance of an internal hydraulic drop whereas for smaller values of h_0/H ($\tilde{<}\ 0.7$) it is a long wave of depression. These experiments served to emphasize the importance of including the effect of the ambient fluid on the gravity current when that current is a sensible fraction of the total depth. Grundy and Rottman [15] in their study of the approach to self-similarity for solutions to the shallow-water equations for sudden releases of fixed volumes, employed a set of model equations that ignored inertial effects due to the presence of the lighter ambient fluid. Their subsequent numerical studies for plane flow show little resemblance to the observed flows [14]. Bonnecaze *et al.* [1] employing a two-step Lax-Wendroff scheme to solve the problem of the sudden release of a fixed volume of homogeneous fluid on a horizontal bottom achieved very good qualitative agreement with the experiments of [14] when they took into account the inertial effects of the ambient fluid. D'Alessio *et al.* [16] using MacCormack's method to integrate numerically the hyperbolic system provided by shallow-water theory were also able to achieve good agreement with the experimental results of [14] for transition to self-similarity when the effects of the ambient fluid were included.

We think it is fair to say, in light of all the material that has been reviewed here and in [4], that we now have a relatively good understanding of compositionally driven gravity flows in rectangular channels in the absence of rotational effects. The areas that possibly merit further study in this restricted class of problems are the larger mathematical issues concerning well-posedness of the related initial boundary value problems. These issues have recently been examined by one of the current authors [17] using the so-called 'principle of localization' [18] on the field equations obtained by freezing the coefficients at fixed constant values of state and, when topography is involved, space variables. Also, according to Simpson [8], there still exists a need for further experimental and theoretical work on the transition zone between steady currents down a slope and time-dependent currents along a horizontal surface. This aspect of gravity currents is presently under investigation [19,20].

Having set out those aspects of compositionally driven gravity currents that figure prominently into the modelling efforts for time-dependent particle-bearing flows we will now look more closely at the progress that has been made in the study of sedimentation from gravity flows.

Attempts at theoretical studies of the overall time-dependent interaction between flow and sedimentation for gravity currents are, in the main, quite recent. The reason for this probably stems from the fact that the prediction of the downstream evolution requires an analysis of the dynamics of the entire finite flow field rather than just the time-dependent behaviour at a point in an otherwise horizontally infinite flow field [21,22]. These studies probably commenced with the works of Sparks *et al.* [5] and Bonnecaze *et al.* [1-3]. In these studies it was assumed that the particles are vertically well-mixed by the turbulence in the current, are advected by the mean flow and settle out through the viscous sublayer at the bottom of the current. It was further assumed that the pressure distribution was hydrostatic and that the horizontal velocity field in the current was vertically uniform. It has subsequently been pointed out that any model based upon these assumptions is inconsistent [4,23]. The detailed arguments presented by Moodie *et al.* [4] show that a consistent shallow-water model can be developed for particle-bearing gravity flows but that it necessarily entails the decoupling of flow and particle dynamics.

We will now extend the model presented in [4] in an attempt to explain the experimentally observed maximum in the areal density of deposit, which occurs at roughly the one-third mark of the total extent of the deposit in many instances. In the experiments of

[1] this was shown to occur at a point roughly 100 cm downstream of the endwall and, as was noted by the authors, neither of their models were capable of predicting this.

The model of [4] is modified to include two important effects that were absent. In the first instance we take into account that the well-mixed fixed-volume suspension behind the lock contains some turbulent energy due to the mixing procedure and that this turbulence initially counteracts the settling of particles when the lock is removed. We argue that the effective velocity with which particles settle out of the dilute suspension is reduced from the Stokes settling velocity by the interaction of the particles with the turbulent field. If the particle size is much larger than the scale of viscous dissipation, it is reasonable to assume that this mechanism is an energy-conserving process. That is, we postulate that turbulence inhibits particle sinking by increasing the mean particle potential energy at the expense of the mean turbulent kinetic energy in the suspending fluid. In this paper, we assume a dilute suspension, so that the back reaction of the particles on the turbulent fluid can be ignored.

Secondly we will incorporate a means to take the experimentally observed particle concentration profiles into account. These profiles show a monotonically increasing concentration of particles with depth below the surface of the particle-bearing bottom current with the near-bottom maximum concentration increasing with particle size or, equivalently, with settling velocity [24].

A detailed derivation of the equations of motion is contained in Section 2, including a justification of the particular choices for the turbulent dissipation time scale and the adopted particle concentration profile. Section 3 is devoted to a numerical investigation of the initial boundary value problems. In the final section we discuss our results and their relation to the experimental observations reported in the literature.

2. Model Development

Consider a gravity current produced by the release of a well-mixed monodisperse suspension of bulk density ρ into an ambient fluid of lesser density ρ_1 overlying a horizontal impervious bottom. The physical configuration is depicted in Figure 1, where $\eta(x,t)$ represents the displacement of the free surface from its undisturbed configuration, and ρ_2 is the density of the interstitial fluid supporting the particles of density ρ_3 $(\rho_3 > \rho_2 > \rho_1)$. $\boldsymbol{u} = (u, w)$ is the fluid velocity in Cartesian coordinates with position vector $\boldsymbol{x} = (x, z)$, H is a vertical length scale, and $h(x,t)$ is the variable thickness of the bottom gravity current. The flow is driven by the buoyancy forces arising because of the difference between the bulk density ρ of the suspension and the density of the ambient fluid, ρ_1. The

Figure 1 goes near here.

density of the suspension is the local volume average of the particle density ρ_3 and the density ρ_2 of the interstitial fluid and is given by

$$\rho(\varphi) = \rho_3\varphi + (1-\varphi)\rho_2, \tag{2.1}$$

where $\varphi = \varphi(x,z,t)$ is the volume fraction occupied by the particles and the suspension is assumed dilute $(0 < \varphi \ll 1)$.

The upper layer (layer 1) is homogeneous so that the density is constant and the corresponding continuity equation is

$$\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z} = 0, \tag{2.2}$$

whereas the continuity equation for the lower layer is

$$\frac{\partial}{\partial t}\left[\rho(\varphi)\right] + \frac{\partial}{\partial x}\left[\rho(\varphi)u_2\right] + \frac{\partial}{\partial z}\left[\rho(\varphi)w_2\right] = 0. \tag{2.3}$$

In our analysis of the unsteady flow of bottom particle-bearing gravity surges we shall assume that the Reynolds numbers are sufficiently large that we may neglect viscous forces in the flow dynamics [1-4]. Flow dynamics are then dominated by the interplay between buoyancy and inertial forces so that the momentum equations for the upper layer may be written as

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + w_1 \frac{\partial u_1}{\partial z} = -\frac{1}{\rho_1} \frac{\partial p_1^*}{\partial x}, \tag{2.4}$$

$$\frac{\partial w_1}{\partial t} + u_1 \frac{\partial w_1}{\partial x} + w_1 \frac{\partial w_1}{\partial z} = -\frac{1}{\rho_1} \frac{\partial p_1^*}{\partial z}, \tag{2.5}$$

where the total pressure field in the upper layer has been written as

$$p_1 = -\rho_1 g z + p_1^* \tag{2.6}$$

with p_1^* the dynamic pressure field and g the acceleration due to gravity. The momentum equations for the lower layer are then, similarly,

$$\rho(\varphi)\left[\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + w_2 \frac{\partial u_2}{\partial z}\right] = -\frac{\partial p_2^*}{\partial x}, \tag{2.7}$$

$$\rho(\varphi)\left[\frac{\partial w_2}{\partial t} + u_2 \frac{\partial w_2}{\partial x} + w_2 \frac{\partial w_2}{\partial z}\right] = -\frac{\partial p_2^*}{\partial z} - \varphi g(\rho_3 - \rho_2), \tag{2.8}$$

where the total pressure field has been transformed as [4]

$$p_2 = \rho_1 g H - \rho_2 g z + p_2^*. \tag{2.9}$$

We require an additional equation describing the conservation of particles in the lower layer. The particle concentration varies throughout the current in a time dependent fashion due to advection and settling. We shall neglect particle entrainment on the assumption that soon after release the fixed-volume suspension has velocities that are insufficient to lift deposited sediment into the current [1-3]. We do not, however, assume throughout the flow regime that turbulent mixing in the current is sufficiently vigorous so as to maintain a vertically uniform particle concentration as was assumed in numerous other works [1-4,25]. Instead, we shall choose vertical concentration profiles that are in accord with recent experimental observations of particle-bearing gravity flows [24]. These observations show a higher concentration of suspended particles at the base of the flow. We assume that the particles leave the current only through the viscous sublayer at the base with a flux $-v_{\text{eff}}\varphi(x,0,t)$, where v_{eff} denotes the effective settling velocity of an isolated particle that is appropriate when the particle concentration is small.

The above assumptions concerning the suspended particles combined with conservation of particles applied to an arbitrary segment of the flow regime $x_2 < x < x_1$ leads directly to

$$\frac{\partial}{\partial t}\Big(\int_0^{h(x,t)} \varphi(x,z,t)dz\Big) + \frac{\partial}{\partial x}\Big(\int_0^{h(x,t)} \varphi(x,z,t)u_2(x,z,t)dz\Big) \\ + \varphi(x,0,t)v_{\text{eff}} = 0. \tag{2.10}$$

Our governing equations now consist of the upper layer equations (2.2), (2.4), (2.5) and the lower layer equations (2.3), (2.7), (2.8), and (2.10) together with the boundary conditions for the free surface, that is,

$$p_1^*(x, H+\eta, t) = \rho_1 g(H+\eta), \tag{2.11}$$

$$w_1(x, H+\eta, t) = \frac{\partial \eta}{\partial t} + u_1(x, H+\eta, t)\,\frac{\partial \eta}{\partial x}\,, \tag{2.12}$$

the boundary conditions for the interface between the two layers,

$$p_2^*(x,h,t) = p_1^*(x,h,t) + \rho_2 gh - \rho_1 g(h+H), \tag{2.13}$$

$$w_i(x,h,t) = \frac{\partial h}{\partial t} + u_i\,\frac{\partial h}{\partial x} \quad (i=1,2), \tag{2.14}$$

and the kinematic bottom boundary condition

$$w_2(x,0,t) = 0. \tag{2.15}$$

We now focus on all of the above equations representing the dynamics of our two-layer system with the goal of developing a set of model equations describing low-aspect ratio flows in each layer with the lower layer comprising a dilute suspension. Although it is clear that the initial flow following release of the well-mixed particle-bearing gravity current of finite volume is a fully three-dimensional unsteady flow, soon after release the current will have spread sufficiently that its length is very much greater than its height. The height will at this stage be slowly varying over the horizontal position x and in time t. To exploit this low-aspect ratio, slowly varying nature of the flow we employ the horizontal and vertical length scales L and H, respectively, and introduce the small parameter $\delta \equiv H/L$ known as the aspect ratio. The dilute nature of the suspension in the lower layer is characterized by the second small parameter $\varepsilon \equiv \varphi_0 \ll 1$ representing the initial spatially uniform volume fraction of particles in the well-mixed release volume.

We now introduce nondimensional and scaled variables according to the following scheme, wherein nondimensional and scaled variables are indicated by a tilde:

$$
\begin{gathered}
x = L\,\widetilde{x}, \quad z = H\,\widetilde{z}, \quad t = \frac{L}{U}\,\widetilde{t}, \quad h = H\,\widetilde{h}, \\
(u_1, u_2) = U(\,\widetilde{u}_1, \widetilde{u}_2\,), \quad (w_1, w_2) = H\,\frac{U}{L}\,(\,\widetilde{w}_1, \widetilde{w}_2), \\
(p_1^*, p_2^*) = U^2(\rho_1 \widetilde{p}_1^{\,*}, \rho_2 \widetilde{p}_2^{\,*}), \quad \eta = \frac{U^2}{g}\,\widetilde{\eta}, \qquad (2.16) \\
\varphi \equiv \varepsilon \widetilde{\varphi}, \quad v_{\text{eff}} \equiv \delta\,\widetilde{v}_{\text{eff}}, \\
g_1' \equiv \frac{\rho_2 - \rho_1}{\rho_2}\,g, \quad g_2' \equiv \frac{\rho_3 - \rho_2}{\rho_2}\,g, \quad U \equiv (g_1' H)^{1/2}.
\end{gathered}
$$

In (2.16), g_1' and g_2' are the reduced gravities and the scaled quantities $\widetilde{\varphi}$ and $\widetilde{v}_{\text{eff}}$ are now $O(1)$ in both ε and δ .

Employing this scheme to nondimensionalize all of our equations gives the following sets:

Upper Layer

$$\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z} = 0, \tag{2.17}$$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + w_1 \frac{\partial u_1}{\partial z} = - \frac{\partial p_1^*}{\partial x}\,, \tag{2.18}$$

$$\delta^2 \Big(\frac{\partial w_1}{\partial t} + u_1 \frac{\partial w_1}{\partial x} + w_1 \frac{\partial w_1}{\partial z} \Big) = - \frac{\partial p_1^*}{\partial z}\,, \tag{2.19}$$

Lower Layer

$$\varepsilon \, \frac{g_2'}{g} \, \Big(\frac{\partial \varphi}{\partial t} + \frac{\partial}{\partial x} \, (\varphi u_2) + \frac{\partial}{\partial z} (\varphi w_2) \Big) + \frac{\partial u_2}{\partial x} + \frac{\partial w_2}{\partial z} = 0, \tag{2.20}$$

$$\begin{aligned} &\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + w_2 \frac{\partial u_2}{\partial z} \\ &\qquad + \varepsilon \varphi \, \frac{g_2'}{g} \, \Big(\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + w_2 \frac{\partial u_2}{\partial z} \Big) = - \frac{\partial p_2^*}{\partial x}\,, \end{aligned} \tag{2.21}$$

$$\begin{aligned} &\delta^2 \Big(\frac{\partial w_2}{\partial t} + u_2 \frac{\partial w_2}{\partial x} + w_2 \frac{\partial w_2}{\partial z} \Big) \\ &\qquad + \varepsilon \delta^2 \, \frac{g_2'}{g} \, \varphi \Big(\frac{\partial w_2}{\partial t} + u_2 \frac{\partial w_2}{\partial x} + w_2 \frac{\partial w_2}{\partial z} \Big) = - \frac{\partial p_2^*}{\partial z} - \frac{\varepsilon \varphi g_2'}{g_1'}\,, \end{aligned} \tag{2.22}$$

Particle Conservation

$$\frac{\partial}{\partial t} \Big(\int_0^h \varphi dz \Big) + \frac{\partial}{\partial x} \Big(\int_0^h \varphi u_2 dz \Big) + \varphi(x, 0, t) \, \frac{v_{\mathrm{eff}}}{U} = 0, \tag{2.23}$$

Free Surface

$$\gamma p_1^*(x, 1 + \gamma\eta, t) = 1 + \gamma\eta, \tag{2.24}$$

$$w_1(x, 1 + \gamma\eta, t) = \gamma \, \frac{\partial \eta}{\partial t} + \gamma u_1(x, 1 + \gamma\eta, t) \, \frac{\partial \eta}{\partial x}\,, \tag{2.25}$$

Interface

$$\gamma p_2^*(x,h,t) = \gamma\,\frac{\rho_1}{\rho_2}\,p_1^*(x,h,t) + h - \frac{\rho_1}{\rho_2}\,(1+h), \tag{2.26}$$

$$w_i(x,h,t) = \frac{\partial h}{\partial t} + u_i\,\frac{\partial h}{\partial x} \quad (i = 1,2), \tag{2.27}$$

Bottom

$$w_2(x,0,t) = 0. \tag{2.28}$$

In the above equations, tildes have been dropped from nondimensional quantities for notational convenience and $\gamma \equiv g_1'/g$ is a nondimensional parameter related to the importance of the free surface in the sense that putting it to zero filters free surface gravity waves out of the model. The other parameters appearing in our nondimensional equations are the aspect ratio δ and the parameter ε measuring the initial volume fraction of suspended particles. This last parameter measures the degree to which suspended particles will contribute to the vertical structure in the velocity field, thereby invalidating shallow water theory. This can be seen in a preliminary fashion simply by examining equation (2.22). This equation shows that even for low aspect ratio flows $(\delta \ll 1)$ that if terms $O(\varepsilon)$ cannot be neglected then

$$\frac{\partial p_2^*}{\partial x} \sim -\frac{g_2'}{g_1'}\varepsilon \int^z \frac{\partial \varphi}{\partial x}\,(x,z',t)dz',$$

which is dependent on z. The horizontal acceleration of the fluid is thus not the same for all fluid particles that are equidistant from $x = 0$ (say) and an assumption of a depth independent horizontal velocity field as required for shallow water theory would be inconsistent [23]. We thus see that even before a detailed analysis is carried out that shallow water theory will be possible only when particles do not enter into the dynamics of the flow.

We now develop our approximate theory based upon the assumption that the initial volume fraction of particles in the well-mixed fixed-volume suspension is small. This assumption is in accord with the experiments reported in [1,6], in which volume fractions were in the range 0.4 to 5.0%. It is aspects of these experiments that we wish to explain so that we will consider $0 < \varepsilon \ll 1$. We also take the aspect ratio to be small with $0 < \varepsilon \sim \delta^2 \ll 1$ in our equations. Under the above assumptions it is reasonable to look for solutions in the form

$$u_i = u_i^{(0)}(x,z,t) + O(\varepsilon), \quad (i = 1,2), \tag{2.29}$$

$$w_i = w_i^{(0)}(x,z,t) + O(\varepsilon), \quad (i = 1,2), \tag{2.30}$$

$$p_i^* = p_i^{(0)}(x,z,t) + O(\varepsilon), \quad (i = 1,2), \tag{2.31}$$

$$\varphi = \varphi^{(0)}(x,z,t) + O(\varepsilon), \tag{2.32}$$

$$\eta = \eta^{(0)}(x,t) + O(\varepsilon), \tag{2.33}$$

$$h = h^{(0)}(x,t) + O(\varepsilon). \tag{2.34}$$

Substituting these expansions into the equations (2.17) - (2.23) and boundary conditions (2.24) - (2.28), extracting the leading-order problems for the two layers, and dropping superscripts from the dependent variables we find:

Upper Layer $O(1)$ Problem

$$\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z} = 0, \tag{2.35}$$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + w_1 \frac{\partial u_1}{\partial z} = - \frac{\partial p_1}{\partial x}\,, \tag{2.36}$$

$$\frac{\partial p_1}{\partial z} = 0, \tag{2.37}$$

$$\gamma p_1(x, 1+\gamma\eta, t) = 1 + \gamma\eta, \tag{2.38}$$

$$w_1(x, 1+\gamma\eta, t) = \gamma \frac{\partial \eta}{\partial t} + \gamma u_1(x, 1+\gamma\eta, t) \frac{\partial \eta}{\partial x}, \tag{2.39}$$

$$\gamma p_2(x, h, t) = \gamma \frac{\rho_1}{\rho_2} p_1(x, h, t) + \gamma h - \frac{\rho_1}{\rho_2}, \tag{2.40}$$

$$w_1(x, h, t) = \frac{\partial h}{\partial t} + u_1 \frac{\partial h}{\partial x}. \tag{2.41}$$

Lower Layer $O(1)$ Problem

$$\frac{\partial u_2}{\partial x} + \frac{\partial w_2}{\partial z} = 0, \tag{2.42}$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + w_2 \frac{\partial u_2}{\partial z} = -\frac{\partial p_2}{\partial x}, \tag{2.43}$$

$$\frac{\partial p_2}{\partial z} = 0, \tag{2.44}$$

$$\gamma p_2(x, h, t) = \gamma \frac{\rho_1}{\rho_2} p_1(x, h, t) + \gamma h - \frac{\rho_1}{\rho_2}, \tag{2.40}$$

$$w_2(x, h, t) = \frac{\partial h}{\partial t} + u_2 \frac{\partial h}{\partial x}, \tag{2.45}$$

$$w_2(x, 0, t) = 0, \tag{2.46}$$

$$\frac{\partial}{\partial t}\Big(\int_0^h \varphi dz\Big) + \frac{\partial}{\partial x}\Big(\int_0^h \varphi u_2 dz\Big) + \varphi(x, 0, t) \frac{v_{\mathrm{eff}}}{U} = 0. \tag{2.47}$$

The leading-order problems for both layers satisfy the sufficiency condition for the development of shallow water theory [4]. After a sequence of formal manipulations we obtain the $O(1)$ governing equations of our coupled two layer particle-bearing system:

$$\frac{\partial}{\partial t} u_1 + \frac{\partial}{\partial x}\Big[\frac{1}{2} u_1^2 + \eta\Big] = 0, \tag{2.48}$$

$$\frac{\partial}{\partial t}(h - \gamma\eta) + \frac{\partial}{\partial x}[(h - \gamma\eta - 1)u_1] = 0, \tag{2.49}$$

$$\frac{\partial u_2}{\partial t} + \frac{\partial}{\partial x}\Big[\frac{1}{2} u_2^2 + h - \gamma\eta + \eta\Big] = 0, \tag{2.50}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu_2) = 0, \tag{2.51}$$

together with equation (2.47).

We now complete our model development by introducing two further modifications to our $O(1)$ theory that will bring its predictions more into line with that which is observed experimentally without adding greatly to the level of complication. In the first instance we present energy arguments for the form of the effective settling velocity that will be used in (2.47). We then relax the unrealistic vertically well-mixed assumption of earlier papers [1-4,6] by introducing shape factors to account for the observed particle concentration profiles [24].

In order to determine the effective settling velocity of the particles in the suddenly released, well-mixed, fixed-volume dilute suspensions we account for the fact that the turbulence initially present will inhibit particle sinking by raising the mean particle potential energy at the expense of the mean turbulent kinetic energy in the suspending fluid. Because we deal solely with dilute suspensions the back reaction of the particles on the turbulent fluid can be ignored.

If the scale of energy injection by external stirring is much larger than that of the suspended particles, the particle size will lie well within the turbulent inertial range. We may then use the transfer rate ϵ of turbulent energy density to smaller scales as a measure of the rate of energy transfer per unit volume to the suspended particles. The well-known self-similarity argument of Kolmogorov [26], which assumes that ϵ is independent of length scale within the inertial range, relates the turbulent energy spectrum $E(k)$, where k is the wavenumber magnitude, to ϵ:

$$E(k) \sim \epsilon^{2/3} k^{-5/3}.$$

It is easily demonstrated from the incompressible Navier–Stokes equation (see Appendix A) that the rate of decay of the total turbulent energy

$$E = \int E(k)\, dk = \frac{1}{2} \int |\boldsymbol{u}|^2 d\boldsymbol{x},$$

is given by

$$\frac{\partial E}{\partial t} = -2\nu Z,$$

where ν is the kinematic viscosity and

$$Z = \frac{1}{2} \int |\boldsymbol{\nabla} \times \boldsymbol{u}|^2 d\boldsymbol{x}$$

is the total enstrophy, or mean-squared vorticity. If we assume that the turbulent spectrum is peaked at a characteristic wavenumber $\bar{k}$ then $Z \approx \bar{k}^2 E$ so that

$$\frac{1}{E} \frac{\partial E}{\partial t} \approx -2\nu \bar{k}^2, \tag{2.52}$$

Now if the turbulence is strong enough, sedimentation will be inhibited until such time as the turbulent energy transfer to the particles balances the potential energy that would be lost due to particle sinking. In typical experimental situations, this balance is achieved well before the lock is opened. We will restrict our attention to this case. In a time τ the total turbulent energy transfer to the particles per unit volume, $\epsilon\tau$, must then balance the buoyant potential energy differential,

$$\epsilon\tau = (\rho_3 - \rho_2) g \zeta,$$

where ζ is the mean physical height difference (relative to Stokes settling) induced by the turbulent interaction. Consequently, the reduction in the settling velocity $v_t = \zeta/\tau$ induced by the turbulence is proportional to ϵ . The departure of the settling velocity from the Stokes value v_s will thus decrease exponentially at the rate δ_* , where

$$\delta_* = -\frac{1}{v_t} \frac{\partial v_t}{\partial t} = -\frac{1}{\epsilon} \frac{\partial \epsilon}{\partial t} = -\frac{3}{2} \frac{1}{E} \frac{\partial E}{\partial t} \approx 3\nu \bar{k}^2. \tag{2.53}$$

If sedimentation just begins to occur as the lock is opened at $t = 0$, the effective settling velocity will then decay exponentially to the Stokes value v_s according to

$$v_{\text{eff}} = v_s - v_t = v_s\big(1 - e^{-\delta_* t}\big). \tag{2.54}$$

We now introduce the second modification to our $O(1)$ theory in order to extend the range of applicability of the model to include the observed structure of sediment concentration profiles in particle-bearing gravity currents. It has been noted [24] that there is an increased concentration with depth in these flows. Previous studies [1-4,6] have always assumed that the particles were vertically well-mixed by turbulence and hence that the particle volume fraction was independent of depth throughout the flow regime. This vertically well-mixed assumption was borrowed from models for time-dependent thermal convection and settling of crystals in magna chambers [27,28] where, because of the continuous convective overturning, it would seem to be more appropriate. We would expect our concentration profiles to be representative of particle-bearing gravity currents in the natural environment, for which the ratios of current velocity to sediment fall velocity are similar to those used here.

We choose as our particle volume fraction the function φ where

$$\varphi(x,z,t) = \overline{\varphi}\,(x,t)\left[1 - \frac{\beta\big(h_0 - h(x,t)\big)z}{h_0 h(x,t)}\right]^{\alpha} \tag{2.55}$$

and $\beta \equiv v_s/(U\delta)$ is the nondimensional scaled Stokes settling velocity with α a numerical exponent that can be chosen to match experimentally determined concentration profiles. We note that if β has a value corresponding to the smallest diameter particles employed in the experiments of Gladstone *et al.*, that is, $\beta \approx 0.02$, then the dependence of φ on depth will be relatively weak and our choice of z-dependent profile gives results for φ that are in close agreement with the vertically well-mixed assumption of previous works [1-4,6]. However, for values of β corresponding to the larger particles, that is, $\beta \approx 0.3 - 0.5$, there is a noticeable depth dependence in the concentration profile, the concavity of which is in accord with the experimentally observed profiles [24]. These issues will be explored in detail in the next section.

Combining (2.47), (2.54), and (2.55) gives for the particle conservation equation

$$\frac{\partial \Phi}{\partial t} + \frac{\partial}{\partial x}(u_2 \Phi) = -\beta\big(1 - e^{-\delta_* t}\big)\,\frac{\Phi}{A(h)}\,, \tag{2.56}$$

where nondimensional variables are employed with tildes omitted, the nondimensional decay rate $\widetilde{\delta}_*$ is given by

$$\widetilde{\delta}_* = \delta_* \beta H / v_s = \frac{\delta_*}{\delta}\sqrt{\frac{H}{g_1'}}, \tag{2.57}$$

and

$$\begin{aligned} \Phi(x,t) &= \overline{\varphi}(x,t)\left(\frac{1}{\alpha+1}\right)\frac{h h_0}{\beta(h_0 - h)}\left[1 - \left(1 - \frac{h_0 - h}{h_0}\,\beta\right)^{\alpha+1}\right] \\ &= \overline{\varphi}(x,t) A(h). \end{aligned} \tag{2.58}$$

By adopting the approach to depth variations in the sediment concentration that is taken here, we are able to work in the realm of shallow water theory with its simplifying depth-independent horizontal velocity field while still accounting for the experimentally observed increase in particle concentration with depth.

In the next section we explore numerically the consequences of our model for the flow and deposition patterns of bottom particle-bearing gravity currents.

3. Numerical Investigation

Numerical solutions to the full set of five model equations (2.48)-(2.51) and (2.56) are obtained using MacCormack's method [29]. This method is an explicit, conservative finite-difference scheme possessing second-order accuracy. Because the scheme is conservative, convergence will be to a physical weak solution of the hyperbolic system of equations (2.48)-(2.51). Further, MacCormack's method provides sharp resolution of shocks and does not require the evaluation of the Jacobian of the flux vector. One drawback is the occurrence of oscillations around the shock. However, these can be adequately damped by applying artificial viscosity. This is done by introducing a numerical diffusion term proposed by Lapidus [30]. This term is of third order and so does not alter the truncation error of MacCormack's method.

Our first set of numerical results are designed to explore the depth-dependent nature of the volume fraction of suspended particles in a typical release of a well-mixed fixed-volume suspension. In Figure 2 we show the depth profile of the bottom gravity current produced by the sudden release of a well-mixed volume with initial fractional depth $h_0 = 0.9$ and base width $x_0 = 0.5$. This figure depicts the particle-bearing current at an elapsed time from release $t = 3.0$ and shows clearly the effect of the reverse flow in the upper fluid layer. This reverse flow, which will be significant if $h_0 \stackrel{\sim}{>} 0.7$, gives rise to the rear shock that can be seen in the figure at $x \approx 1.25$. This issue concerning bore

Figure 2 goes near here.

formation and the depth-dependent nature of this phenomenon has been dealt with at length in two earlier publications [4,16] and so will not be discussed at length here. All of our fixed-volume releases analyzed here will have the same dimensions.

In Figure 3 we show clearly how, according to our assumption on the depth dependence of the volume fraction, this volume fraction will vary over the depth for two different values of the nondimensional scaled Stokes settling velocity β. These two values represent small $(\beta = 0.05)$ and larger $(\beta = 0.5)$ particles and are typical of

Figure 3 goes near here.

values that arise in the experimental work reported in [1-3], where particle size varies from 23 to $200\,\mu m$. We also note, with reference to Figure 2, that the depth profiles for the volume fractions that are displayed in Figure 3 are for the station at $x = 1.5$, $t = 3$ and so are for the deepest part of the gravity surge. The other parameter that is required for our calculation is the nondimensional decay rate defined in (2.57). Assuming that the suspending fluid is water, for which $\nu \approx 0.01\,\mathrm{cm}^2/\mathrm{s}$, the total depth H of fluid in the tank is $30\,\mathrm{cm}$ and $\delta = 0.1$, we find that typically $\widetilde{\delta}_* \approx 0.235\bar{k}^2\,\mathrm{cm}^2$. If we further assume that the characteristic wave number $\bar{k}$ at which the turbulent spectrum is peaked corresponds to wavelengths in the range $8 - 12\,\mathrm{cm}$, we find that typical values for the decay rate $\widetilde{\delta}_*$ will lie in the range $0.06 - 0.14$. Our chosen values for the peak of the turbulent spectrum are consistent with the physical dimension of the experimental apparatus used in the conduct of typical fixed-volume release experiments [31] in which the well-mixed suspension behind the lock is created by a hand-held stirring device.

Figure 3 shows that the larger particles for which $\beta = 0.5$ accumulate near the bottom so that the volume fraction has considerable variation over the depth of the current. With the smaller particles for which $\beta = 0.05$ we see that there is very little variation with depth in the volume fraction so that our model for the depth variation of the volume fraction gives results that are very close to the assumed vertically well-mixed particles of previous studies [1-5] when particles settle more slowly. This is in accord with the experimental observations reported recently by Garcia [24] and we feel that this approach gives considerable improvement over the approach that simply assumes there is no vertical variation in particle concentration.

The next pair of figures show some interesting results of our detailed calculations and point out one of the important aspects of our model. In both of these figures $\beta = 0.5$

Figures 4 and 5 go near here.

and $\delta_* = 0.1$ and the values of α range from 1.5 to 3.5. Figure 4 depicts the variation in volume fraction with depth at a point $x = 0.5$, which we can see from Figure 2 is a station where the particle-bearing current has a depth that is roughly 10% of its initial depth. We also note that at this station the current is relatively low in particle concentration when compared to the concentrations that are computed for the station at $x = 1.5$ where the current is relatively deep. The computations for $x = 1.5$ are displayed in Figure 5 and show the variation with depth in the volume fraction. In contrast to the results depicted in Figure 4 we see here that the volume fraction can be as high as 70% of its initial value whereas from Figure 4 the computed volume fraction in the rear of the current has fallen to below 20% of its initial value. These results reflect the fact that the source term in the particle conservation equation (2.56) is inversely proportional to the depth of the current, via the factor $A(h)$, and so particles sediment most rapidly from the rear part of the current, which has become thin owing to shock formation. This occurrence of a relatively particle poor zone at the rear of the particle-bearing current and a relatively particle rich zone in the head has been noted in previous modelling attempts [1-3]. The vertical variations in particle densities that are provided by our model are, however, new to the literature and do provide a better fit to reality.

In Figures 6 and 7 we have plotted the volume fraction of particles at $x = 0.5$ and $x = 1.5$ for particles for which $\beta = 0.05$. Hence these plots are for particles that settle considerably more slowly than did those of Figures 4 and 5. This is reflected in the higher values of the volume fractions at both stations with the lower values still being associated with the region in which the current is thinnest. However, with this value of β the volume fraction remains at over 80% of its initial value even at the upper boundary of the particle-bearing gravity current.

Figures 6 and 7 go near here.

In the figures that follow we plot the results of our computation to determine the areal density of deposit on the bottom after the sedimentation process has been completed and all particles have been deposited. The nondimensional density of deposit is given by

$$m(x,t) = \int_0^t \varphi(x,0,t')\beta(1-e^{-\delta_* t'})dt', \tag{3.1}$$

wherein all variables are nondimensional and we have adhered to our convention of omitting tildes from such quantities.

In Figures 8 and 9 we have plotted the nondimensional areal density of deposit versus x for particles with differing settling numbers β and for two different values of the parameter α that enters into our depth-dependent volume fraction of particles. On

$\boxed{\text{Figures 8 and 9 go near here.}}$

comparing the results presented in these two figures we see that the ultimate depositional pattern is relatively insensitive to the choice of this parameter even over a wide range of settling numbers. In both of these figures we have chosen to set δ_* to the value 0.1 that, for reasons presented earlier, we believe to be in the correct regime.

In both of these figures the behaviour is as seen in the experiments of Gladstone *et al.* [6]. As the settling parameter β is increased the particles get preferentially deposited in the proximal zone. The trends that are displayed in these two figures closely resemble the experimentally determined deposit density variations with distance that are displayed in Figure 5 of Gladstone *et al.* [6].

In Figures 10 and 11 below we have examined the dependence of deposit density on the parameter α in our model. It is clear from these figures that for $\beta = 0.05$ the

$\boxed{\text{Figures 10 and 11 go near here.}}$

dependence of areal density on α is very weak over a wide range of values. When β is increased to a value corresponding to larger particles with their corresponding higher fall rate we see that the areal density of deposit will be slightly more sensitive to the choice of α but that its lateral extent is influenced little by changes in α.

In Figures 12 and 13 we display the areal density of deposit for two different values of the Stokes settling parameter β over a wide range of values for the decay rate δ_*. The

$\boxed{\text{Figures 12 and 13 go near here.}}$

first difference that we observe between the results displayed in these two figures is the effect of a larger value of β on the settling pattern. With a larger β the particles are preferentially deposited in the proximal zone.

The dependence of the areal density of deposit on the decay rate δ_* shows clearly that decreasing the value of this parameter allows particles to remain in suspension over a larger time period thereby increasing the range over which particles are deposited. These smaller values of δ_* correspond to a turbulent field with a greater lifespan. This persistence of the turbulence could be due to a decreased kinematic viscosity or changes in the scale of the characteristic wavenumber at which the turbulent spectrum is peaked or combinations of these mechanisms.

Discussion

In this article we have presented a theory for monodisperse, noneroding particle-suspension gravity currents produced by the sudden release of well-mixed fixed-volume suspensions. This study was undertaken in order to explain certain discrepancies between theory and experiment for such releases, the most noteworthy of these being the difference between computed and experimentally observed areal densities of deposits. The experimentally determined deposits always exhibited a characteristic depression in the areal density in the region adjacent to the lockgate. By accounting for the presence of turbulent energy in the fluid volume behind the lockgate we have been able to obtain results that are in close accord with the experimental results that have been reported over the past few years [1-3,5,6]. We are not aware of any other theoretical model that is capable of closing the gap between prediction and experimental observation (nor were the authors of the above reports).

The other modification that was introduced was the concept of a shape function for the vertical variations in particle concentration. Although this modification does not have a great influence on the computed long-time areal density of deposits it does effect the instantaneous particle concentration in a way that is in accord with the scant observations on vertical distribution of particles [24].

The predictions of our theoretical model show that the sedimentation patterns of particle-bearing gravity currents are strongly influenced by the size of suspended particles. We saw from Figure 8 and 9 that the density of deposit for each particle size exhibits a peak point. Deposit density then decreases gradually with distance beyond this point.

The deposit produced by a suspension of relatively coarse particles only has the greatest density close to the lockgate. The deposit density profile produced by a suspension carrying fine particles alone is very different as can be seen from Figures 8 and 9. Although of similar form, the deposit density profile with distance shows the lowest maximum, farther from the lockgate, and shallower gradients. The fine particles have effective fall velocities sufficiently small that many particles remain suspended over a longer time scale. These conclusions based on our model computations are in complete agreement with the experimental observations reported in the literature. (See [6] in particular.)

We would conclude that this local maximum in areal density of deposit may be explained entirely by the experimental conditions that obtain at the time of release of the fixed-volume suspensions. In all of the experiments, as stated clearly in the published work, the "particles were well mixed in the lock and the gate was quickly lifted to release the current" [3]. Under these conditions the release volume will contain significant turbulent energy at the instant of release. If one were to argue against this mechanism by claiming that the observed local maximum is due to particles that have already settled being transported downstream in the form of a bedload by the flow then, since smaller particles are more easily transported, the magnitude of the maximum should increase with decreasing particle size. Since this is not observed in practice it is more likely that our 'initial turbulence' argument is the correct one.

Appendix A: Turbulent Energy Decay Rate

Here we derive the rate of decay of kinetic energy for three-dimensional incompressible Boussinesq (constant density) turbulence described by the Navier–Stokes equation,

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} u = \nu \nabla^2 \boldsymbol{u} - \frac{\boldsymbol{\nabla} P}{\rho}. \tag{A.1}$$

The assumption of constant density allows us to write A.1 in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{1}{2}\boldsymbol{\nabla} u^2 - \boldsymbol{u} \times (\boldsymbol{\nabla} \times \boldsymbol{u}) = \nu \nabla^2 \boldsymbol{u} - \boldsymbol{\nabla}\left(\frac{P}{\rho}\right).$$

Upon taking the dot product with the velocity $\boldsymbol{u}$ and integrating with respect to $\boldsymbol{x}$ over the entire flow, we find, assuming that $\boldsymbol{u}$ vanishes at the boundaries,

$$\begin{aligned}\frac{\partial}{\partial t}\int \frac{u^2}{2}\, d\boldsymbol{x} &= -\int \boldsymbol{u} \cdot \boldsymbol{\nabla}\left(\frac{u^2}{2} + \frac{P}{\rho}\right) d\boldsymbol{x} + \nu \int \boldsymbol{u} \cdot \nabla^2 \boldsymbol{u}\, d\boldsymbol{x} \\ &= \int \left(\frac{u^2}{2} + \frac{P}{\rho}\right) \boldsymbol{\nabla} \cdot \boldsymbol{u}\, d\boldsymbol{x} + \nu \int \boldsymbol{u} \cdot \left[\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \boldsymbol{\nabla} \times \boldsymbol{\omega}\right] d\boldsymbol{x},\end{aligned}$$

where we have introduced the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$. Under the incompressibility assumption $\nabla \cdot \boldsymbol{u} = 0$, we find

$$\frac{\partial}{\partial t}\int \frac{u^2}{2}\, d\boldsymbol{x} = -\nu \int \boldsymbol{u} \cdot \boldsymbol{\nabla} \times \boldsymbol{\omega}\, d\boldsymbol{x} = \nu \int [\boldsymbol{\nabla} \cdot (\boldsymbol{u} \times \boldsymbol{\omega}) - \boldsymbol{\omega} \cdot \boldsymbol{\nabla} \times \boldsymbol{u}]\, d\boldsymbol{x} = -\nu \int |\boldsymbol{\omega}|^2\, d\boldsymbol{x}.$$

The inclusion of a gravitational force (or any other force derivable from a potential) in (A.1) leads to the same conclusion.

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Figure Captions

FIGURE 1. Geometry of the two-layer model used in this paper.

FIGURE 2. The gravity current at $t = 3$ for the case $h_0 = 0.9$, $x_0 = 0.5$, and $g'/g = 0.05$, with $\Delta x = 0.05$.

FIGURE 3. The vertical structure of the particle concentration profiles for various values of β at $x = 1.5$ and $t = 3$ with $\alpha = 3.5$ and $\delta_* = 0.1$.

FIGURE 4. The vertical structure of the particle concentration profiles for various values of α at $x = 0.5$ and $t = 3$ with $\beta = 0.5$ and $\delta_* = 0.1$.

FIGURE 5. The vertical structure of the particle concentration profiles for various values of α at $x = 1.5$ and $t = 3$ with $\beta = 0.5$ and $\delta_* = 0.1$.

FIGURE 6. The vertical structure of the particle concentration profiles for various values of α at $x = 0.5$ and $t = 3$ with $\beta = 0.05$ and $\delta_* = 0.1$.

FIGURE 7. The vertical structure of the particle concentration profiles for various values of α at $x = 1.5$ and $t = 3$ with $\beta = 0.05$ and $\delta_* = 0.1$.

FIGURE 8. Nondimensional density of deposit as a function of x for various values of β with $\alpha = 2.0$ and $\delta_* = 0.1$.

FIGURE 9. Nondimensional density of deposit as a function of x for various values of β with $\alpha = 3.5$ and $\delta_* = 0.1$.

FIGURE 10. Nondimensional density of deposit as a function of x for various values of α with $\beta = 0.05$ and $\delta_* = 0.1$.

FIGURE 11. Nondimensional density of deposit as a function of x for various values of α with $\beta = 0.5$ and $\delta_* = 0.1$.

FIGURE 12. Nondimensional density of deposit as a function of x for various values of δ_* with $\beta = 0.05$ and $\alpha = 2.0$.

FIGURE 13. Nondimensional density of deposit as a function of x for various values of δ_* with $\beta = 0.5$ and $\alpha = 2.0$.