The 3D Asymptote Generalization of MetaPost Bézier Interpolation

John C. Bowman[*1]

[1] Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, T6G 2G1 Canada

The descriptive vector graphics language `Asymptote` provides for the typesetting of mathematical figures a capability analogous to what TEX provides for typesetting equations. Labels are typeset with TEX, for professional quality and overall document consistency, and the simplex linear programming method is used to solve overall size constraint issues between fixed-sized and scalable objects. Asymptote's three-dimensional generalization of the spline interpolation algorithms of Hobby (*Discrete and Computational Geometry* **1**, 1986) is shape invariant under three-dimensional affine transformations and reduces in the planar case to Hobby's prescription. By using recursive refinement to add additional Bézier control points, one can then efficiently approximate a perspective-invariant nonuniform rational B-spline.

The descriptive graphics language `Asymptote`[1] generates native Postscript and PDF output, the portable standard format used for professional printing. `Asymptote` was inspired by Hobby's `MetaPost` (a modified version of `MetaFont`, the program that Knuth wrote to produce the TEX fonts), but features a more elegant C++-like syntax, high-order functions, and robust floating-point numerics.

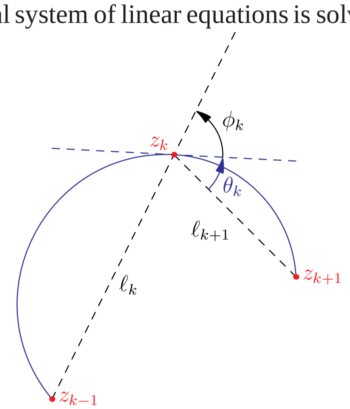
Asymptote also generalizes to three dimensions Hobby's two-dimensional algorithms for drawing an aesthetically pleasing, numerically efficient interpolating spline, given a set of nodes and optional tangent directions [1]. Unfortunately, Postscript and PDF support only Bézier splines, which are invariant under orthographic (affine) but not perspective projection. Since non-uniform rational B-splines, which *are* invariant under perspective projection, cannot be reduced to Bézier splines without loss, the error introduced by using three-dimensional cubic Bézier splines for perspective drawings must be examined.

In any dimension, applying an affine transformation $x_i' = A_{ij}x_j + C_i$ to a cubic Bézier curve $x(t) = \sum_{k=0}^{3} B_k(t)P_k$ for $t \in [0,1]$, where $B_k(t)$ is the kth cubic Bernstein polynomial yields the Bézier curve

$$x_i'(t) = \sum_{k=0}^{3} B_k(t)A_{ij}(P_k)_j + C_i = \sum_{k=0}^{3} B_k(t)P_k',$$

in terms of the transformed $k^{\rm th}$ control point P_k', noting that $\sum_{k=0}^{3} B_k(t) = 1$. Thus if the projection from three to two dimensions is orthographic, the only issue is how to determine unspecified three-dimensional control points.

In two dimensions, Asymptote fills in unspecified Bézier control points using the algorithms of Hobby and Knuth [1, 2]. First, a tridiagonal system of linear equations is solved to determine any unspecified directions θ_k and ϕ_k through each node z_k:



$$\frac{\theta_{k-1} - 2\phi_k}{\ell_k} = \frac{\phi_{k+1} - 2\theta_k}{\ell_{k+1}}.$$

The resulting shape may be adjusted by modifying optional *tension* parameters and *curl* boundary conditions.

Having prescribed outgoing and incoming path directions $e^{i\theta}$ at node z_0 and $e^{i\phi}$ at node z_1 relative to the vector $z_1 - z_0$, the control points are then determined as:

$$u = z_0 + e^{i\theta}(z_1 - z_0)f(\theta, -\phi),$$
$$v = z_1 - e^{i\phi}(z_1 - z_0)f(-\phi, \theta),$$

where the relative distance function $f(\theta, \phi)$ is given by Hobby [1986]:

$$f(\theta,\phi) = \frac{2 + \sqrt{2}\,(\sin\theta - \frac{1}{16}\sin\phi)(\sin\phi - \frac{1}{16}\sin\theta)(\cos\theta - \cos\phi)}{3\left(1 + \frac{1}{2}(\sqrt{5} - 1)\cos\theta + \frac{1}{2}(3 - \sqrt{5}\,)\cos\phi\right)}.$$

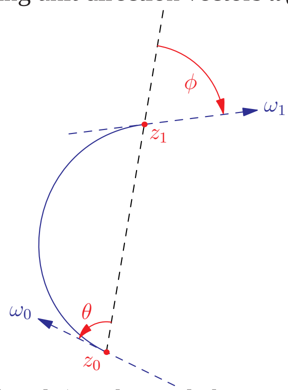
[*] Corresponding author: e-mail: bowman@math.ualberta.ca, Phone: +1 780 492 0532, Fax: +1 780 492 6826

[1] Available under the GNU General Public License from `http://asymptote.sourceforge.net/`

1 Three-dimensional generalization of Hobby's algorithms

We require that the 3D generalization of Hobby's algorithm reduces to the 2D algorithm in the planar case. Any unknown incoming or outgoing tangency directions are first determined by applying Hobby's direction algorithm in the plane containing z_{k-1}, z_k, and z_{k+1}. The only ambiguity that can arise is in the overall sign of the angles, which relates to viewing each local 2D plane from opposing normal directions. A reference vector constructed from the mean unit normal of successive segments can be used to resolve such ambiguities.

An algorithm for determining the control points u and v follows on expressing Hobby's algorithm in terms of the absolute incoming and outgoing unit direction vectors ω_0 and ω_1, respectively:



$$u = z_0 + \omega_0 \, |z_1 - z_0| \, f(\theta, -\phi),$$

$$v = z_1 - \omega_1 \, |z_1 - z_0| \, f(-\phi, \theta),$$

where we interpret θ and ϕ as the angle between the corresponding path direction vector and $z_1 - z_0$. In this case there is an unambiguous reference vector for determining the relative sign of the angles θ and ϕ.

2 Perspective projection

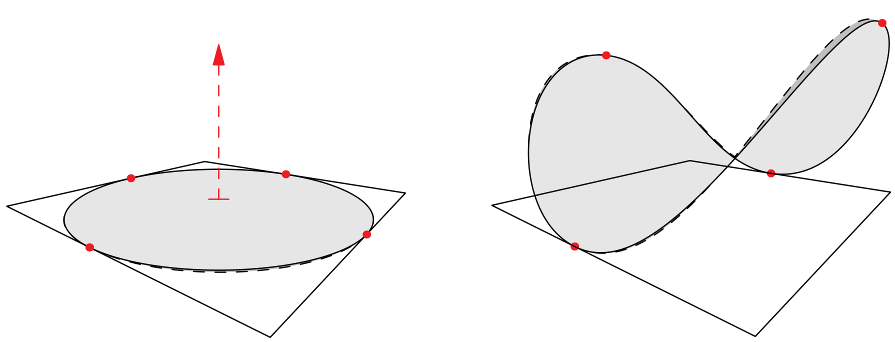
In `Asymptote`, a unit circle in the X–Y plane can be expressed with the three-dimensional path syntax:

```
(1,0,0)..(0,1,0)..(-1,0,0)..(0,-1,0)..cycle
```

and then distorted into a saddle:

```
(1,0,0)..(0,1,1)..(-1,0,0)..(0,-1,1)..cycle.
```

If these paths are visualized with a perspective projection, the result of simply projecting the nodes and control points of the three-dimensional Bézier curves to two dimensional Bézier curves is indicated below by the dashed path. Fortunately, as will be discussed in future work, the exact projection of the equivalent nonuniform rational B-spline, indicated by the solid path, can be efficiently approximated as a two-dimensional Bézier curve with additional nodes and control points.



Acknowledgements The author gratefully acknowledges Andy Hammerlindl, Tom Prince, and many others for their contributions to the design and development of Asymptote, L. Nobre and Troy Henderson for suggesting techniques for approximating nonuniform rational B-splines, and the Natural Sciences and Engineering Research Council of Canada, the Pacific Institute for Mathematical Sciences, and the University of Alberta Faculty of Science for their financial support.

References

[1] John D. Hobby. Smooth, easy to compute interpolating splines. *Discrete Comput. Geom.*, 1:123–140, 1986.
[2] Donald E. Knuth. *The METAFONTbook*. Addison-Wesley, Reading, Massachusetts, 1986.