Math 655: Statistical Theories of Turbulence

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Chapter 1

Turbulence in Two and Three Dimensions

1.A Introduction

"Turbulence is the last great unsolved problem of classical physics." [1]

The problem of turbulence, particularly the problem of predicting the drag on a body moving through a turbulent fluid, has occupied the attention of scientists for centuries. Even today, this problem remains one of the most elusive yet fascinating unsolved puzzles of science. There are vital interests, both scientific and commercial, in understanding and reducing the detrimental effects of enhanced turbulent drag or diffusion in a wide range of applications. Our modern interests in energy-efficient transportation, in accurate weather forecasting, in the prediction of large-scale ocean movements, in global climate modeling, and in magnetic fusion attach great importance to the problem of turbulence. Despite the intense effort that, over the years, has been devoted to an understanding of turbulence, only limited progress has been made toward the development of a satisfactory mathematical theory.

For a flow with sufficiently small velocities, the motion is *laminar* and can be readily analyzed by perturbation analysis of the Navier–Stokes equations. However, if the flow involves high velocities, the motion becomes *turbulent*, or highly chaotic. While the Navier–Stokes equations provide, in principle, an adequate model for the behaviour of such a fluid, a new difficulty enters. One finds that the solution of these equations for this case requires so many Fourier harmonics (or other orthogonal basis functions) that the problem becomes intractable, both analytically and numerically. For example, Orszag [1970] estimates that solution of highly developed turbulence can require on the order of 10^{20} numerical operations!

[1] Holmes, P., J. L. Lumley and G. Berkooz, "Turbulence, Coherent Structures, Dynamical Systems, and Symmetry," Cambridge University Press (1996) and references therein.

Thus, to this day much of our quantitative knowledge of turbulence is empirical. The experimental database is limited since it is difficult to make the necessary measurements even in controlled laboratory systems, let alone in the turbulent systems of the real world. Turbulence experiments are particularly hard to repeat with the same initial conditions. Until we have a better understanding, it is also not clear which parameters are important and which measurements should be made. Numerical simulation of turbulent systems has become a popular alternative to costly experiments since the advent of the supercomputer, but for the foreseeable future the resolution of even these advanced machines will be insufficient to discern the fine-scale features of fully developed turbulence.

Even the highly idealized problem of homogeneous isotropic incompressible turbulence continues to defy both analytical and numerical efforts. Here, one of the principle questions is to characterize the distribution of turbulent energy among various scales in the flow. A particularly intriguing challenge is to derive, directly from the equations of motion, the power-law scaling of the inertial-range energy spectrum that was obtained on dimensional grounds by Kolmogorov [1941] and has since been the subject of extensive numerical and experimental scrutiny.

1.B Incompressible Homogeneous Turbulence

A field is *homogeneous* if its statistical properties are independent of x, so that there is no preferred origin. There can then be no walls; the domain is unbounded (a periodic domain is also possible). Let $\boldsymbol{u}(\boldsymbol{x},t)$ be the velocity field of the fluid, $\rho(\boldsymbol{x},t)$ be the fluid density, $P(\boldsymbol{x},t)$ be the pressure, ν be the kinematic viscosity (assumed constant), and $\boldsymbol{F}(\boldsymbol{x},t)$ be an homogeneous external stirring force (per unit mass). The equations for incompressible homogeneous turbulence are then

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla}P + \nu\nabla^2\boldsymbol{u} + \boldsymbol{F}, \tag{1.1}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0, \tag{1.2}$$

$$\frac{\partial \rho}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\rho = 0, \tag{1.3}$$

$$\boldsymbol{\nabla}\rho(\boldsymbol{x},0) = 0 \quad \forall\boldsymbol{x}. \tag{1.4}$$

The first equation is the Navier–Stokes equation. The continuity equation, Eq. (1.2), can be used to reduce the incompressibility equation, Eq. (1.3), to the perhaps more familiar form $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$. Recall that for a C^1 vector field $\boldsymbol{u}$ on a simply connected domain

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = \boldsymbol{0} \iff \boldsymbol{u} = \boldsymbol{\nabla}\times\boldsymbol{A}'$$

for some C^2 *vector potential* $\boldsymbol{A}'$. The vector potential is unique only to within the gradient of some scalar function. It is often convenient to transform to the *Coulomb gauge* by defining

$$\boldsymbol{A} \doteq \boldsymbol{A}' + \boldsymbol{\nabla}\phi,$$

where ϕ satisfies the Poisson equation

$$\nabla^2\phi = -\boldsymbol{\nabla\cdot A}',$$

so that $\boldsymbol{\nabla\cdot A} = \boldsymbol{\nabla\cdot A}' + \nabla^2\phi = 0$ (we emphasize definitions with the notation $\doteq$). Hence we can restate Eq. (1.B) as

$$\boldsymbol{\nabla\cdot u} = \boldsymbol{0} \iff \boldsymbol{u} = \boldsymbol{\nabla}\times\boldsymbol{A} \quad \text{with} \quad \boldsymbol{\nabla\cdot A} = 0.$$

Equations (1.3) and(1.4) imply that

$$\frac{\partial\rho}{\partial t} = \boldsymbol{\nabla}\rho = 0 \quad \forall t \geq 0.$$

Without loss of generality, we choose units for mass such that $\rho = 1$. Upon taking the divergence of Eq. (1.1), we obtain an equation for the pressure P:

$$\boldsymbol{\nabla\cdot}[\boldsymbol{F} - (\boldsymbol{u\cdot\nabla})\boldsymbol{u}] = \nabla^2 P. \tag{1.5}$$

Given suitable boundary conditions, this Poisson equation can then be solved for the pressure P.[2]

Alternatively, one may eliminate P from the problem entirely by taking the curl of Eq. (1.1). It is helpful to first use the identity

$$(\boldsymbol{u\cdot\nabla})\boldsymbol{u} = \frac{1}{2}\boldsymbol{\nabla}u^2 - \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u})$$

to rewrite Eq. (1.1) as

$$\frac{\partial\boldsymbol{u}}{\partial t} + \frac{1}{2}\boldsymbol{\nabla}u^2 - \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) = -\boldsymbol{\nabla}P + \nu\nabla^2\boldsymbol{u} + \boldsymbol{F}. \tag{1.6}$$

It is convenient to introduce the *vorticity* $\boldsymbol{\omega} \doteq \boldsymbol{\nabla}\times\boldsymbol{u}$, a measure of the amount of rotation in the flow. Upon exploiting the fact that both $\boldsymbol{u}$ and $\boldsymbol{\omega}$ are solenoidal fields ($\boldsymbol{\nabla\cdot u} = \boldsymbol{\nabla\cdot\omega} = 0$), we can express

$$\begin{aligned}\boldsymbol{\nabla}\times((\boldsymbol{u\cdot})\boldsymbol{\nabla u}) &= \boldsymbol{\nabla}\times\left[\frac{1}{2}\boldsymbol{\nabla}u^2 - \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u})\right]\\ &= -\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{\omega})\\ &= \boldsymbol{u\cdot\nabla\omega} - \boldsymbol{\omega\cdot\nabla u}.\end{aligned}$$

The curl of Eq. (1.6) thus simplifies to

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + (\boldsymbol{u\cdot\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega\cdot\nabla})\boldsymbol{u} + \nu\nabla^2\boldsymbol{\omega} + \boldsymbol{\nabla}\times\boldsymbol{F}. \tag{1.7}$$

[2] In particular, for periodic boundary conditions, it is easy to verify the *solvability* condition that the spatial integral of the left-hand side vanishes. This remark also applies to Eq. (1.B).

1.C Symmetries

Equation (1.1) has the following symmetries:

$$\text{Space-translation: } (t, \boldsymbol{r}, \boldsymbol{u}) \to (t, \boldsymbol{r} + \boldsymbol{\rho}, \boldsymbol{u}), \qquad \boldsymbol{\rho} \in \mathbb{R}^3;$$

$$\text{Time-translation: } (t, \boldsymbol{r}, \boldsymbol{u}) \to (t + \tau, \boldsymbol{r}, \boldsymbol{u}), \qquad \tau \in \mathbb{R};$$

$$\text{Galilean transformation: } (t, \boldsymbol{r}, \boldsymbol{u}) \to (t, \boldsymbol{r} + \boldsymbol{U}t, \boldsymbol{u} + \boldsymbol{U}), \qquad \boldsymbol{U} \in \mathbb{R}^3;$$

$$\text{Parity: } (t, \boldsymbol{r}, \boldsymbol{u}) \to (t, -\boldsymbol{r}, -\boldsymbol{u});$$

$$\text{Rotation: } (t, \boldsymbol{r}, \boldsymbol{u}) \to (t, \boldsymbol{A}\boldsymbol{r}, \boldsymbol{A}\boldsymbol{u}), \qquad \boldsymbol{A} \in SO(\mathbb{R}^3);$$

$$\text{Gradient: } P(\boldsymbol{r}, t) \to P(\boldsymbol{r}, t) + G(t);$$

$$\text{Reynolds: } (t, \boldsymbol{r}, \boldsymbol{u}, \nu) \to \left(\gamma t, \lambda \boldsymbol{r}, \frac{\lambda}{\gamma}\boldsymbol{u}, \frac{\lambda^2}{\gamma}\nu\right), \qquad \lambda, \gamma \in \mathbb{R}, \gamma \neq 0. \tag{1.8a}$$

The final symmetry preserves the value of the dimensionless parameter

$$R = \frac{ur}{\nu},$$

known as the *Reynolds number*, and encapsulates the famous *similarity principle* of fluid dynamics. Strictly speaking, for $\gamma \neq \lambda^2$, it is a symmetry of the turbulent medium and not of the differential equation *per se* since it rescales the viscosity parameter in addition to the independent and dependent variables. For $\lambda > 0$, defining $h = 1 - \frac{\log\gamma}{\log\lambda}$ allows us to rewrite $\lambda/\gamma = \lambda^h$, so that the Reynolds symmetry may be equivalently expressed as

$$(t, \boldsymbol{r}, \boldsymbol{u}, \nu) \to \left(\lambda^{1-h}t, \lambda\boldsymbol{r}, \lambda^h\boldsymbol{u}, \lambda^{1+h}\nu\right), \qquad \lambda \in \mathbb{R}_+, h \in \mathbb{R}.$$

Note that for the particular case $h = -1$, the viscosity remains unchanged.

1.D Conservation Laws

When the flow is inviscid ($\nu = 0$) and forcing is absent ($\boldsymbol{F} = \boldsymbol{0}$), Eq. (1.1) conserves the *energy* (mean-squared velocity)

$$E \doteq \frac{1}{2} \int u^2 \, d\boldsymbol{x}.$$

To see this, consider the time derivative of the integral of the dot product of Eq. (1.6) with $\boldsymbol{u}$:

$$\begin{aligned} \frac{dE}{dt} &= \int \boldsymbol{u}\boldsymbol{\cdot}\frac{\partial \boldsymbol{u}}{\partial t}\, d\boldsymbol{x} \\ &= -\int \boldsymbol{u}\boldsymbol{\cdot}\left[\boldsymbol{\nabla}\left(\frac{u^2}{2}+P\right)+\boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u})+\nu\nabla^2\boldsymbol{u}+\boldsymbol{F}\right] d\boldsymbol{x} \\ &= \int\left(\frac{u^2}{2}+P\right)\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u}\, d\boldsymbol{x}+\nu\int \boldsymbol{u}\boldsymbol{\cdot}\nabla^2\boldsymbol{u}\, d\boldsymbol{x}+\int \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{F}\, d\boldsymbol{x}, \end{aligned} \tag{1.9}$$

assuming zero boundary conditions at infinity (or periodic boundary conditions). We thus see that the global energy is conserved when $\nu = 0$ and $\boldsymbol{F} = \boldsymbol{0}$. Moreover, when $\nu \neq 0$ we can express the decay rate of energy for unforced turbulence as

$$\begin{aligned} \frac{dE}{dt} &= \nu \int \boldsymbol{u}\boldsymbol{\cdot}\nabla^2\boldsymbol{u}\, d\boldsymbol{x}, \\ &= -\nu \int \boldsymbol{u}\boldsymbol{\cdot}\left[\boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u})-\nabla^2\boldsymbol{u}\right] d\boldsymbol{x} \\ &= -\nu \int \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{u})\, d\boldsymbol{x} \\ &= -\nu \int \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\times\omega\, d\boldsymbol{x} \\ &= \nu \int \boldsymbol{\nabla}\boldsymbol{\cdot}(\boldsymbol{u}\times\boldsymbol{\omega})-\omega\boldsymbol{\cdot}\boldsymbol{\nabla}\times\boldsymbol{u}\, d\boldsymbol{x} \\ &= -\nu \int \omega^2\, d\boldsymbol{x} \\ &= -2\nu Z, \end{aligned} \tag{1.10}$$

in terms of the *enstrophy* (mean-squared vorticity),

$$Z \doteq \frac{1}{2} \int \omega^2 \, d\boldsymbol{x}.$$

The energy is an inviscid invariant in both two and three dimensional turbulence. In two dimensions the enstrophy Z is also an invariant: if $\boldsymbol{u} = (u, v, 0)$ and is independent of the z coordinate, it follows that $\boldsymbol{\omega} = \omega\hat{\boldsymbol{z}}$ and the *vortex-stretching* term $\boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{u}$ in Eq. (1.7) will vanish, so that

$$\frac{\partial\omega}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\omega = 0. \tag{1.11}$$

Then

$$\begin{aligned}\frac{dZ}{dt} &= \int \omega\frac{\partial\omega}{\partial t}\,d\boldsymbol{x}\\ &= -\int (\boldsymbol{u}\cdot\boldsymbol{\nabla})\left(\frac{w^2}{2}\right)d\boldsymbol{x}\\ &= \int \left(\frac{\omega^2}{2}\right)\boldsymbol{\nabla}\cdot\boldsymbol{u}\,d\boldsymbol{x}\\ &= 0.\end{aligned}$$

In fact in two dimensions there exists uncountably many other invariants, known as *Casimir invariants*, of the inviscid equations. Any continuously differentiable function of the (scalar) vorticity is conserved by Eq. (1.11):

$$\begin{aligned}\frac{d}{dt}\int f(\omega)\,d\boldsymbol{x} &= \int f'(\omega)\frac{\partial\omega}{\partial t}\,d\boldsymbol{x} = -\int f'(\omega)(\boldsymbol{u}\cdot\boldsymbol{\nabla})\omega\,d\boldsymbol{x}\\ &= -\int (\boldsymbol{u}\cdot\boldsymbol{\nabla})f(\omega)\,d\boldsymbol{x} = \int f(\omega)\boldsymbol{\nabla}\cdot\boldsymbol{u}\,d\boldsymbol{x} = 0.\end{aligned} \tag{1.12}$$

However, it is believed that only the positive-semidefinite quadratic invariants (energy and enstrophy) play a fundamental role in the turbulent dynamics.

Another important property of two-dimensional turbulence is that it may be cast in terms of a single scalar field. The vorticity $\boldsymbol{\omega} = \omega\hat{\boldsymbol{z}}$ is related to the vector potential $\boldsymbol{A} = (A_x, A_y, A_z)$ in the Coulomb gauge by

$$\begin{aligned}\omega\hat{\boldsymbol{z}} &= \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})\\ &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A}\\ &= -\nabla^2\boldsymbol{A}.\end{aligned} \tag{1.13}$$

Hence $\nabla^2 A_x = \nabla^2 A_y = 0$. Given periodic or infinite boundary conditions one may then without loss of generality take $A_x = A_y = 0$ so that $\boldsymbol{A}$, like $\boldsymbol{\omega}$, has only one component, in the direction normal to the plane of motion. It is conventional to define $\psi \doteq -A_z$ to be the *stream function*. Thus

$$\begin{aligned}\boldsymbol{u} &= \boldsymbol{\nabla}\times\boldsymbol{A}\\ &= \hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\psi.\end{aligned}$$

We see from Eq. (1.13) that the stream function is related to the vorticity by $\omega = \nabla^2\psi$. In two dimensions, Eq. (1.7) thus simplifies to

$$\frac{\partial\nabla^2\psi}{\partial t} + (\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla})\nabla^2\psi = \nu\nabla^4\psi + \hat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times\boldsymbol{F}. \tag{1.14}$$

As mentioned above, the reason that enstrophy is conserved in two dimensions is that $\boldsymbol{u}$ and $\boldsymbol{\omega}$ are always perpendicular. This implies that the mean *helicity*

$$H \doteq \frac{1}{2}\int \boldsymbol{u}\cdot\boldsymbol{\omega}\, d\boldsymbol{x}$$

vanishes in two dimensions. In three dimensions, the total helicity may have a nonzero value, but that value is still conserved by the inviscid dynamics. We leave the proof of this statement as an exercise. Although the helicity is a quadratic invariant, it differs from the energy and enstrophy in that it can be zero or negative in a turbulent flow.

1.E The Energy Spectrum

The energy spectrum of fully developed homogeneous turbulence is thought to be composed of three distinct wavenumber regions: a region of energy injection (by the force $\boldsymbol{F}$ in Eq. (1.1) at the largest scales, an intermediate *inertial range* characterized by zero forcing and zero dissipation, and, at the very smallest scales, a region dominated by viscosity. In 1941, Kolmogorov proposed his famous $k^{-5/3}$ scaling law for the inertial-range energy spectrum of homogeneous and isotropic three-dimensional turbulence. This conjecture is believed to apply only when the forcing and dissipation scales are widely separated.

The distribution of energy with scale is conveniently studied by introducing the integral Fourier transform from the n-dimensional spatial coordinate $\boldsymbol{x}$ to the n-dimensional wavenumber $\boldsymbol{k}$:

$$\boldsymbol{u}_{\boldsymbol{k}} = \int \boldsymbol{u}(\boldsymbol{x}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, d\boldsymbol{x}.$$

Note that when $\boldsymbol{u}$ is a real field, Eq. (1.E) implies that $\boldsymbol{u}^*_{-\boldsymbol{k}} = \boldsymbol{u}_{\boldsymbol{k}}$.

We can recover the original field $\boldsymbol{u}(\boldsymbol{x})$ from $\boldsymbol{u}_{\boldsymbol{k}}$ using the inverse Fourier Transform

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{1}{(2\pi)^n}\int \boldsymbol{u}_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, d\boldsymbol{k}.$$

Another way of expressing this relationship is the identity

$$\int e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\, d\boldsymbol{k} = (2\pi)^n \delta(\boldsymbol{x}-\boldsymbol{x}').$$

In order to study the distribution of energy over various scales, we will soon want to transform Eq. (1.1) and Eq. (1.7) with the aid of the Fourier Convolution Theorem:

$$\begin{aligned}\int f(\boldsymbol{x})g(\boldsymbol{x})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\,d\boldsymbol{x} &= \frac{1}{(2\pi)^{2n}}\int\int f_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\,d\boldsymbol{p}\int g_{\boldsymbol{q}}e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\,d\boldsymbol{q}\,e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\,d\boldsymbol{x}\\ &= \frac{1}{(2\pi)^{2n}}\int\int f_{\boldsymbol{p}}g_{\boldsymbol{q}}\int e^{i(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k})\cdot\boldsymbol{x}}\,d\boldsymbol{x}\,d\boldsymbol{p}\,d\boldsymbol{q}\\ &= \frac{1}{(2\pi)^{n}}\int\int f_{\boldsymbol{p}}g_{\boldsymbol{q}}\delta(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k})\,d\boldsymbol{p}\,d\boldsymbol{q}\\ &= \frac{1}{(2\pi)^{n}}\int f_{\boldsymbol{p}}g_{\boldsymbol{k}-\boldsymbol{p}}\,d\boldsymbol{p}. \end{aligned} \tag{1.15}$$

For identical real fields f and g, Eq. (1.15) reduces to Parseval's Theorem when $\boldsymbol{k}=0$:

$$\begin{aligned}\int f^2(\boldsymbol{x})\,d\boldsymbol{x} &= \frac{1}{(2\pi)^n}\int f_{\boldsymbol{p}}f_{-\boldsymbol{p}}\,d\boldsymbol{p}\\ &= \frac{1}{(2\pi)^n}\int |f_{\boldsymbol{k}}|^2\,d\boldsymbol{k},\end{aligned}$$

which may be used to express the total energy E in n-dimensional Fourier space:

$$\begin{aligned}E &= \frac{1}{2}\int \boldsymbol{u}^2\,d\boldsymbol{x}\\ &= \frac{1}{2}\frac{1}{(2\pi)^n}\int |\boldsymbol{u}_{\boldsymbol{k}}|^2\,d\boldsymbol{k}\\ &= \int_0^\infty E(k)\,dk,\end{aligned}$$

where in three dimensions the *energy spectrum* $E(k)$ is given by

$$E(k) = \frac{1}{2}\frac{1}{(2\pi)^3}\int_0^{2\pi}\int_0^{\pi} |\boldsymbol{u}_{\boldsymbol{k}}|^2\,k^2\sin\theta\,d\theta\,d\phi.$$

In any dimension, the total energy is simply the area under the graph of $E(k)$ with respect to k.

It is worth mentioning that *pseudospectral* numerical simulations of turbulence in a periodic domain use the N-point discrete Fourier transform

$$\hat{\boldsymbol{u}}_k = \sum_{j=0}^{N-1}\boldsymbol{u}_j e^{-\frac{2\pi ijk}{N}}$$

which has the inverse

$$\boldsymbol{u}_j = \frac{1}{N}\sum_{k=0}^{N-1}\hat{\boldsymbol{u}}_k e^{\frac{2\pi ijk}{N}}.$$

The orthogonality relationship underlying this transform pair is elucidated on substituting $z = e^{2\pi i(j+\bar{j})/N}$:

$$\sum_{k=0}^{N-1} e^{2\pi i(j+\bar{j})k/N} = \sum_{k=0}^{N-1} z^k = \begin{cases} N & \text{if } j+\bar{j} = mN \text{ for } m \in \mathbb{Z}, \\ \frac{1-z^N}{1-z} = 0 & \text{otherwise.} \end{cases}$$

1.F 3D Inertial Range

"Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity." [Richardson, 1922]

In 1941, Kolmogorov conjectured that the inertial-range energy spectrum of three-dimensional turbulence exhibits a power-law scaling of the form

$$E(k) = C\epsilon^{\alpha} k^{\beta}, \tag{1.16}$$

where C is a univeral constant, known as the Kolmogorov constant.

In the simplest formulation of this dimensional argument, one observes that the energy spectrum $E(k)$ has units of energy times length (L^3/T^2); noting that one must integrate over k to get the total energy. The *energy injection rate* ϵ has units of energy per unit time: L^2/T^3. Balancing

$$\left[\frac{L^3}{T^2}\right] = \left[\frac{L^2}{T^3}\right]^{\alpha} \left[\frac{1}{L}\right]^{\beta}$$

is then equivalent to requiring that $3 = 2\alpha - b$ and $2 = 3a$, from which we deduce that $\alpha = 2/3$ and $b = -5/3$:

$$E(k) = \epsilon^{2/3} k^{-5/3}. \tag{1.17}$$

There are several problems with this dimensional reasoning. For example, it is not clear that k is the only spatial scale that should enter Eq. (1.16). Perhaps the largest and smallest scales in the inertial range (as well as the box size L in a bounded domain of size L^n) could also play a role. The smallest scale in the inertial range, the dissipation wavenumber k_d, clearly depends on the value of the viscosity.

Kolmogorov assumed that in the limit of zero viscosity, the inertial-range energy spectrum does not depend on k_d. Equation (1.16) can be extended to the dissipation range in terms of a (dimensionless) function f (which he conjectured to be universal):

$$E(k) = \epsilon^{2/3} k^{-5/3} f\left(\frac{k}{k_d}\right),$$

where $\lim_{k\to 0} f(k/k_d) > 0$.

We can use the Kolmogorov spectrum to derive an expression for the dissipation wavenumber k_d. First, we recall from Eq. (1.10) that the energy dissipation is $2\nu Z$. It is convenient to use Parseval's theorem to express the enstrophy Z in terms of the energy spectrum $E(k)$. On noting that the incompressibility condition $\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{u}_{\boldsymbol{k}} = 0$ implies that $|\boldsymbol{\omega}_{\boldsymbol{k}}|^2 = |\boldsymbol{k}\times\boldsymbol{u}_{\boldsymbol{k}}|^2 = k^2 |u_{\boldsymbol{k}}|^2$, we see that

$$Z = \frac{1}{2}\int |\boldsymbol{k}\times\boldsymbol{u}_{\boldsymbol{k}}|^2 \, d\boldsymbol{k} = \frac{1}{2}\int k^2 |u_{\boldsymbol{k}}|^2 \, d\boldsymbol{k} = \int k^2 E(k)\, dk.$$

Kolmogorov defined the dissipation wavenumber k_d for steady-state turbulence as the scale where the energy dissipation balances the energy injection ϵ:

$$\epsilon = \nu \int_0^{k_d} k^2 E(k)\, dk$$

On substituting Eq. (1.17), this balance becomes

$$\epsilon = \nu C \int_0^{k_d} \epsilon^{2/3} k^{1/3}\, dk.$$

One finds

$$\epsilon^{1/3} \propto \nu k_d^{4/3},$$

so that

$$k_d \propto \left(\frac{\epsilon}{\nu^3}\right)^{1/4}. \tag{1.18}$$

Denote the energy dissipation as a function of ν by $\epsilon(\nu)$. Kolmogorov assumed that $\lim\limits_{\nu\to 0} \epsilon(\nu) = \epsilon_0 > 0$, so that $\lim\limits_{\nu\to 0} k_d = \infty$. This is equivalent to requiring that

$$\lim_{\nu\to 0} f\left(\frac{k}{\kappa_d}\right) = f(0)$$

exists.

However, it is also possible that f is singular in this limit; i.e. for some $\zeta > 0$

$$\lim_{x\to 0} x^\zeta f(x) = \text{const} > 0.$$

This alternative admits the possibility of *intermittency*.

Another problem with Equation (1.17) is that it is intended to describe only the asymptotic limit of the energy spectrum (of infinite-Reynold's number turublence) as $k \to \infty$. We now present some phenomenological arguments that suggest how the Kolmogorov spectrum can be matched up to the large-scale dynamics.

- Turbulence consists of a sea of *eddies*, or energy disturbances, of many sizes.
- Small eddies are more numerous than large eddies.

- The large eddies will significantly distort the small eddies.
- *Random interactions* of the many small eddies on the large eddies tend to *average out* their distorting effect on the large eddies.
- Energy is not created or destroyed within the inertial range; it is merely *redistributed* among the inertial-range wavenumbers.
- The total energy in all eddies larger than k^{-1} is $\int_0^k E(\bar{k})\, d\bar{k}$, where $E(k)$ is the *energy spectrum.*
- Energy transfer rate from large eddies to eddies of size k^{-1} [and energy $kE(k)$] is

$$\Pi(k) \doteq \eta(k)\, kE(k),$$

where the *eddy turnover rate* $\eta(k)$ is the rate at which a unit amount of energy is transferred.

Dimensional analysis:

$$\eta^2 \sim \int_0^k p^2 E(p)\, dp$$

$$i.e. \qquad \left(\frac{1}{t}\right)^2 \sim \left(\frac{1}{\ell}\right)^2 \left(\frac{\ell}{t}\right)^2. \tag{1.19}$$

- Thus, energy transfer rate is proportional to

$$\bar{\Pi}(k) \doteq \left[\int_0^k \bar{k}^2 E(\bar{k})\, d\bar{k}\right]^{1/2} kE(k).$$

The constant of proportionality is related to the Kolmogorov constant. Kolmogorov [1941]: significant interactions between the turbulent eddies are *local* in wavenumber space.

- Very large eddies will not interact *directly* with very small eddies, but only *via* eddies of an intermediate size.
- For stationary turbulence, Kolmogorov's locality hypothesis $\Rightarrow$ $\bar{\Pi}$ independent of k.

  Denote $f(k) = kE(k)$. Differentiate the identity

$$\frac{\bar{\Pi}^2}{f^2(k)} = \int_0^k \bar{k} f(\bar{k})\, d\bar{k}$$

with respect to k to obtain

$$-2\bar{\Pi}^2 \frac{f'}{f^4} = k.$$

Integrate this result between k_0 and k, where k_0 is the smallest wavenumber in the inertial range $\Rightarrow$

$$E(k) = k^{-1}\left[\frac{3}{4\bar{\Pi}^2}(k^2 - k_0^2) + k_0^{-3} E^{-3}(k_0)\right]^{-1/3} \qquad (k \geq k_0).$$

This modified Kolmogorov law [Bowman 1996] may be rewritten as

$$E(k) = \left(\frac{4}{3}\right)^{1/3} \bar{\Pi}^{2/3} k^{-5/3} \chi^{-1/3}(k) \qquad (k \geq k_0),$$

in terms of the correction factor

$$\chi(k) \doteq 1 - \frac{k_0^2}{k^2}(1 - \chi_0),$$

where $\chi_0 \doteq 4\bar{\Pi}^2 k_0^{-5} E^{-3}(k_0)/3 = \chi(k_0) > 0$.

For $k \gg k_0 |1 - \chi_0|^{1/2}$, we obtain the usual Kolmogorov scaling

$$E(k) = \left(\frac{4}{3}\right)^{1/3} \bar{\Pi}^{2/3} k^{-5/3}.$$

Three regimes:

- If $\chi_0 < 1$, energy spectrum will be *steeper* than $k^{-5/3}$ near k_0.
- If $\chi_0 \approx 1$, energy spectrum will scale as $k^{-5/3}$.
- If $\chi_0 > 1$, energy spectrum will be *shallower* than $k^{-5/3}$ near k_0.

1.G 2D Enstrophy Inertial Range

Previous arguments were based on conservation of

$$E = \int_0^\infty E(k)\, dk.$$

- Turbulence in two dimensions is complicated by the presence of an additional *enstrophy* invariant:

$$Z = \int_0^\infty k^2 E(k)\, dk.$$

- Kolmogorov's picture of energy transfer to the smallest scales can no longer be correct.
- Such a redistribution of the energy ⇒ *creation* of new enstrophy since $Z(k) = k^2 E(k)$.
- Kraichnan [1967], Kraichnan [1971a]: *enstrophy* transfer rate is independent of k. Enstrophy transfer rate is proportional to

$$\bar{\Pi}_Z(k) \doteq \left[\int_0^k \bar{k}^2 E(\bar{k})\, d\bar{k}\right]^{1/2} k^3 E(k).$$

Let $f(k) = k^3 E(k)$. Differentiate with respect to k:

$$-2\bar{\Pi}^2 \frac{f'}{f^4} = \frac{1}{k}.$$

Let k_1 be the smallest wavenumber in the inertial range. Integrate between k_1 and k to obtain

$$E(k) = k^{-3} \left[\frac{3}{2\bar{\Pi}_Z^2} \log\left(\frac{k}{k_1}\right) + k_1^{-9} E^{-3}(k_1) \right]^{-1/3} \qquad (k \geq k_1).$$

Rewrite as

$$E(k) = \left(\frac{2}{3}\right)^{\frac{1}{3}} \bar{\Pi}_Z^{2/3} k^{-3} \chi^{-1/3}(k) \qquad (k \geq k_1),$$

where

$$\chi(k) \doteq \log\left(\frac{k}{k_1}\right) + \chi_1$$

and $\chi_1 \doteq 2\bar{\Pi}_Z^2 k_1^{-9} E^{-3}(k_1)/3 = \chi(k_1) > 0$.

- Since $\chi_1 > 0$, there is no divergence at $k = k_1$, in contrast to Kraichnan's result:

$$E(k) \sim k^{-3} \left[\log\left(\frac{k}{k_1}\right) \right]^{-1/3}.$$

- The logarithmic factor will be significant when $\chi_1 \ll 1$ and for wavenumbers near k_1.

1.H 2D Energy Inertial Range

In two dimensions, Fjørtoft[1953] demonstrated that the energy cannot cascade downscale as Kolmogorov had argued in the case of three-dimensional turbulence. Fjørtoft's argument is based on the fact that nonlinear term $(\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\psi\boldsymbol{\cdot}\boldsymbol{\nabla})\nabla^2\psi$ in Eq. (1.14) conserves both the energy

$$\begin{aligned}\frac{1}{2}\int \left|\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\psi\right|^2 d\boldsymbol{x} &= \frac{1}{2}\int \left|\boldsymbol{\nabla}\psi\right|^2 d\boldsymbol{x} \\ &= \frac{1}{2}\left(\frac{1}{2\pi}\right)^n \int k^2 \left|\psi_{\boldsymbol{k}}\right|^2 d\boldsymbol{k} \\ &\doteq \int E(k)\, dk\end{aligned}$$

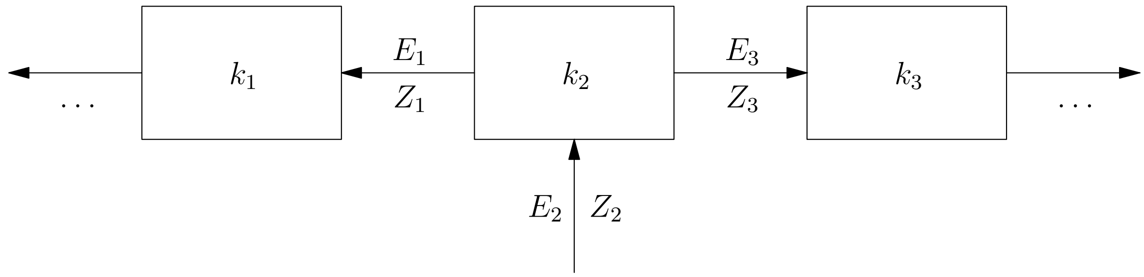

Figure 1.1: Energy and enstrophy transfers in two-dimensional turbulence.

and enstrophy

$$\begin{aligned}\frac{1}{2}\int |\boldsymbol{\omega}|^2 \, d\boldsymbol{x} &= \frac{1}{2}\int \left|\nabla^2\psi\right|^2 \, d\boldsymbol{x} \\ &= \frac{1}{2}\left(\frac{1}{2\pi}\right)^n \int k^4 \left|\psi_{\boldsymbol{k}}\right|^2 \, d\boldsymbol{k} \\ &\doteq \int Z(k)\, dk.\end{aligned}$$

Note that $Z(k) = k^2 E(k)$. Now partition the inertial range into three consecutive narrow wavenumber bins containing characteristic wavenumber magnitudes k_1, k_2, and k_3, with $k_1 < k_2 < k_3$, as illustrated in Fig. 1.1. We obtain the following balance of energy and enstrophy for the middle wavenumber bin, under the assumption that the turbulence is local:

$$E_2 = E_1 + E_3, \tag{1.20}$$

$$Z_2 = Z_1 + Z_3. \tag{1.21}$$

Since $Z_i \approx k_i^2 E_i$ for $i = 1, 2, 3$, Eq. (1.21) becomes

$$k_2^2 E_2 \approx k_1^2 E_1 + k_3^2 E_3. \tag{1.22}$$

Upon solving Eqs. (1.20) and(1.22) we find that

$$E_1 \approx \frac{k_3^2 - k_2^2}{k_3^2 - k_1^2} E_2,$$

$$E_3 \approx \frac{k_2^2 - k_1^2}{k_3^2 - k_2^2} E_2.$$

For example, when $k_1 = k$, $k_2 = 2k$, and $k_3 = 4k$, we find that

$$E_1 \approx \frac{4}{5}E_2, \qquad Z_1 \approx \frac{1}{5}Z_2,$$

$$E_3 \approx \frac{1}{5}E_2, \qquad Z_3 \approx \frac{4}{5}Z_2.$$

We see in this example that energy cascades to large scales and enstrophy cascades to small scales. The upscale transfer of energy occurs at a uniform rate; consequently, an energy inertial range of the form

$$E(k) \sim k^{-5/3}\chi^{-1/3}(k)$$

will develop at the large scales.

If we force at an intermediate wavenumber k_1, with $k_0 < k_1 < k_d$, where k_d is a characteristic dissipation wavenumber, a dual cascade will result:

$$E(k) \sim \begin{cases} k^{-5/3} & \text{if } k_0 \le k \le k_1 \\ k^{-3} & \text{if } k_1 \le k \le k_d \end{cases}$$

- Q: If energy cascades to large scales, where does it go, if there is no large-scale dissipation?

1.I Energy Balance

How does Eq. (1.1) appear in Fourier space? Using Eq. (1.5) to eliminate the pressure and denoting the nonlinear terms by $\boldsymbol{S_k}$, we may write Eq. (1.1) as

$$\frac{\partial \boldsymbol{u_k}}{\partial t} + \nu k^2 \boldsymbol{u_k} = \boldsymbol{S_k} + \left(\boldsymbol{1} - \frac{\boldsymbol{kk}}{k^2}\right)\boldsymbol{F}.$$

Upon multiplying by $\boldsymbol{u}^*_{\boldsymbol{k}}$ and integrating over wavenumber angle, we obtain an equation for $E(k)$ of the form

$$\frac{\partial E(k)}{\partial t} + 2\nu k^2 E(k) = 2T(k) + G(k), \tag{1.23}$$

where $T(k)$ arises from the nonlinearity and $G(k)$ arises from the forcing. Let

$$\Pi(k) \doteq 2\int_k^\infty T(\bar{k})\, d\bar{k} \tag{1.24}$$

represent the transfer of energy by the nonlinear terms into the wavenumber magnitude region $[k,\infty)$. Upon integrating Eq. (1.23) from k to ∞ we obtain the energy balance equation

$$\frac{\partial}{\partial t}\int_k^\infty E(\bar{k})\, d\bar{k} = \Pi(k) - \epsilon(k),$$

where

$$\epsilon(k) \doteq 2\nu \int_k^\infty \bar{k}^2 E(\bar{k})\, d\bar{k} - \int_k^\infty G(\bar{k})\, d\bar{k}$$

is the total transfer, via dissipation and forcing, *out* of wavenumbers higher than k.

- A positive (negative) value for $\Pi(k)$ represents a flow of energy to wavenumbers higher (lower) than k.
- From Eq. (1.9) we find that when $\nu = 0$ and $\boldsymbol{F} = 0$,

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \int_0^\infty E(\bar{k})\, d\bar{k} \\ &= 2 \int_0^\infty T(\bar{k})\, d\bar{k}, \end{aligned}$$

so that we may rewrite Eq. (1.24) as

$$\Pi(k) = -2 \int_0^k T(\bar{k})\, d\bar{k}.$$

- Note that $\Pi(0) = \Pi(\infty) = 0$.
- In a steady state, the left-hand side of Eq. (1.I) vanishes and $\Pi(k) = \epsilon(k)$. This serves as an excellent numerical diagnostic for when a steady state has been reached.

1.J Statistical Equipartition

In two dimensions, we have seen that the vortex stretching term $(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{u}$ vanishes in the vorticity equation Eq. (1.7), and the vorticity vector $\boldsymbol{\omega} = \omega\hat{\boldsymbol{z}}$ is normal to the plane of motion. The equation for the scalar vorticity ω is

$$\frac{\partial \omega}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\omega = \nu\nabla^2\omega + f,$$

where $f = \hat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times\boldsymbol{F}$ and $\boldsymbol{u} = \hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\nabla^{-2}\boldsymbol{\omega}$.

1.K Statistical Closures

Statistical closures constitute an intriguing alternative to conventional numerical simulations of the primitive dynamical equations of turbulence. The Navier–Stokes equation at high Reynolds number, for example, defies direct numerical computation, primarily because the solutions of this strongly nonlinear equation vary rapidly in both space and time. In contrast, statistical closures provide approximate descriptions of the *average* behavior of an *ensemble* of turbulent realizations; these statistical solutions are relatively smooth.

The construction of a statistical description of turbulence is far from unambiguous. The averaging of a nonlinear equation leads to an infinite hierarchy of moment equations that is usually closed by adopting some approximate relation between high-order moments and low-order moments.

We begin with the fundamental equation

$$\boxed{\left(\frac{\partial}{\partial t}+\nu_{\boldsymbol{k}}\right)\psi_{\boldsymbol{k}}(t)=\frac{1}{2}\int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\, d\boldsymbol{q}\, M_{\boldsymbol{kpq}}\psi_{\boldsymbol{p}}{}^{*}\psi_{\boldsymbol{q}}{}^{*},}$$

where $\int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\, d\boldsymbol{q} \doteq \int d\boldsymbol{p}\, d\boldsymbol{q}\, \delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})$. The mode-coupling coefficients $M_{\boldsymbol{kpq}}$ can always be symmetrized so that

$$M_{\boldsymbol{kpq}}=M_{\boldsymbol{kqp}}.$$

They also satisfy one more more symmetries of the form

$$\sigma_{\boldsymbol{k}}M_{\boldsymbol{kpq}}+\sigma_{\boldsymbol{p}}M_{\boldsymbol{pqk}}+\sigma_{\boldsymbol{q}}M_{\boldsymbol{qkp}}=0,$$

where $\sigma_{\boldsymbol{k}}$ are real time-independent factors. For example in two dimensions, if ψ represents the stream function, the symmetrized mode-coupling coefficients

$$M_{\boldsymbol{kpq}}=\frac{\hat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{p}\boldsymbol{\times}\boldsymbol{q}}{k^2}(q^2-p^2)$$

satisfy the symmetry Eq. (1.K) for both $\sigma_{\boldsymbol{k}}=k^2$ and $\sigma_{\boldsymbol{k}}=k^4$, leading directly to the conservation of energy and enstrophy, respectively. An important property of mode-coupling coefficients (for most applications) is that

$$M_{\boldsymbol{kpq}}=0 \text{ whenever } \boldsymbol{p}=\pm\boldsymbol{q} \text{ or } p=q.$$

It is convenient to define the two-time correlation function

$$C_{\boldsymbol{k}}(t,t')\doteq\langle\psi_{\boldsymbol{k}}(t)\psi_{\boldsymbol{k}}{}^{*}(t')\rangle\,.$$

The energy $E\doteq\frac{1}{2}\sum_{\boldsymbol{k}}\left\langle|\psi_{\boldsymbol{k}}|^2\right\rangle$ may be defined in terms of the equal-time correlation function

$$\begin{aligned}C_{\boldsymbol{k}}(t)&\doteq C_{\boldsymbol{k}}(t,t)\\&=\left\langle|\psi_{\boldsymbol{k}}(t)|^2\right\rangle.\end{aligned}$$

The nonlinear Green's function $R_{\boldsymbol{k}}(t,t')$ is the infinitesimal response to a source function $\bar{\eta}_{\boldsymbol{k}}$ added to the right-hand side of Eq. (1.K):

$$R_{\boldsymbol{k}}(t,t')\doteq\left\langle\frac{\delta\psi_{\boldsymbol{k}}(t)}{\delta\bar{\eta}_{\boldsymbol{k}}(t')}\right\rangle\bigg|_{\bar{\eta}_{\boldsymbol{k}}=0}.$$

Here is a schematic illustration of the construction of a statistical closure for a prototype nonlinear equation in the random variable $\psi \doteq \psi(t)$:

$$\frac{\partial\psi}{\partial t} + \nu\psi = M\psi\psi.$$

- Second moment:

$$\begin{aligned}\frac{\partial \left\langle\psi\psi\right\rangle}{\partial t} &= 2\left\langle \frac{\partial}{\partial t}\psi\psi \right\rangle \\ &= -2\nu\left\langle\psi\psi\right\rangle + 2M\left\langle\psi\psi\psi\right\rangle.\end{aligned}$$

- The *normal* (Gaussian) approximation $\left\langle\psi\psi\psi\right\rangle = 0$ amounts to *linear theory*!
- Instead, formulate the equation for $\left\langle\psi\psi\psi\right\rangle$:

$$\frac{\partial}{\partial t}\left\langle\psi\psi\psi\right\rangle + 3\nu\left\langle\psi\psi\psi\right\rangle = 3M\left\langle\psi\psi\psi\psi\right\rangle.$$

If we prescribe Gaussian initial conditions at $t = 0$, then we can solve this equation to obtain

$$\left\langle\psi\psi\psi\right\rangle = 3M\int_0^t d\bar{t}\, e^{-3\nu(t-\bar{t})}\left\langle\bar{\psi}\bar{\psi}\bar{\psi}\bar{\psi}\right\rangle,$$

where $\bar{\psi} \doteq \psi(\bar{t})$.

- Make the *quasinormal* approximation:

$$\left\langle\bar{\psi}\bar{\psi}\bar{\psi}\bar{\psi}\right\rangle = 3\left\langle\bar{\psi}\bar{\psi}\right\rangle\left\langle\bar{\psi}\bar{\psi}\right\rangle$$

For Gaussian statistics, this holds *exactly*.

- We arrive at the quasinormal closure:

$$\boxed{\frac{\partial}{\partial t}\left\langle\psi\psi\right\rangle + 2\nu\left\langle\psi\psi\right\rangle = 18MM\int_0^t d\bar{t}\, e^{-3\nu(t-\bar{t})}\left\langle\bar{\psi}\bar{\psi}\right\rangle\left\langle\bar{\psi}\bar{\psi}\right\rangle.}$$

- Ogura [1963], Orszag [1977] demonstrated that the quasinormal closure can incorrectly predict *negative energies*!
- A superior closure is obtained by *renormalizing*.
- The effect of this renormalization is to replace the *unperturbed* (linear) Green's function

$$R^{(0)}(t,\bar{t}) \equiv e^{-3\nu(t-\bar{t})}H(t-\bar{t}),$$

where H is the Heaviside step function, by the statistical mean R of the *perturbed* (nonlinear) Green's function $\top R$, which satisfies

$$\frac{\partial}{\partial t}\top R + \nu\top R - 2M\psi\top R = \delta(t-\bar{t}).$$

- The equation for $C \doteq \langle\psi\psi\rangle$ then takes on the form

$$\frac{\partial}{\partial t}C + 2\nu C = 18MM\int_0^t d\bar{t}\, R\, C\, C.$$

- One obtains a similar equation for R:

$$\frac{\partial}{\partial t}R + \nu R = 9MM\int_0^t d\bar{t}\, R\, C\, R + \delta(t-\bar{t}).$$

1.K.1 General form of a closure:

$$\left(\frac{\partial}{\partial t} + \nu_{\boldsymbol{k}}\right)C_{\boldsymbol{k}}(t,t') + \overbrace{\int_0^t d\bar{t}\,\Sigma_{\boldsymbol{k}}(t,\bar{t})C_{\boldsymbol{k}}(\bar{t},t')}^{\text{nonlinear (eddy) damping}} = \underbrace{\int_0^{t'} d\bar{t}\,\mathcal{F}_{\boldsymbol{k}}(t,\bar{t})R^*_{\boldsymbol{k}}(t',\bar{t})}_{\text{nonlinear noise}}, \tag{1.25}$$

$$\left(\frac{\partial}{\partial t} + \nu_{\boldsymbol{k}}\right)R_{\boldsymbol{k}}(t,t') + \int_{t'}^t d\bar{t}\,\Sigma_{\boldsymbol{k}}(t,\bar{t})R_{\boldsymbol{k}}(\bar{t},t') = \delta(t-t'),$$

1.L Direct-interaction approximation (DIA)

$$\Sigma_{\boldsymbol{k}}(t,\bar{t}) = -\int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\, d\boldsymbol{q} M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} R^*_{\boldsymbol{p}}(t,\bar{t}) C^*_{\boldsymbol{q}}(t,\bar{t}).$$

$$\mathcal{F}_{\boldsymbol{k}}(t,\bar{t}) = \frac{1}{2}\int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\, d\boldsymbol{q} M_{\boldsymbol{kpq}} M^*_{\boldsymbol{kpq}} C^*_{\boldsymbol{p}}(t,\bar{t}) C^*_{\boldsymbol{q}}(t,\bar{t}),$$

Advantages of the DIA [Kraichnan 1958], [Kraichnan 1959], [Kraichnan 1961], [Leslie 1973], [Krommes 1984]

- Reduces correctly to perturbation theory.
- Handles self-consistency in a reasonable way.
- Produces two-time spectral information.
- Can be obtained from an underlying *stochastic model.*

  Disadvantages:
- Contains time-history integrals, so is nontrivial to compute.
- In 3D, predicts an energy inertial range $E(k) \sim k^{-3/2}$ instead of $k^{-5/3}$.
- In 2D, predicts an enstrophy inertial range $E(k) \sim k^{-5/2}$ instead of k^{-3}.
- Only handles second-order statistics; mistreats higher-order coherent structures.

1.L.1 Inertial-range scaling of the DIA

The equal-time DIA covariance equation may be written in the compact form

$$\frac{\partial}{\partial t} C_{\boldsymbol{k}}(t) + 2\,\mathrm{Re}\, N_{\boldsymbol{k}}(t) = 2F_{\boldsymbol{k}}(t),$$

where

$$N_{\boldsymbol{k}}(t) \doteq \nu_{\boldsymbol{k}} C_{\boldsymbol{k}}(t) - \int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\, d\boldsymbol{q}\, M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} \bar{\Theta}^*_{\boldsymbol{pqk}}(t),$$

$$F_{\boldsymbol{k}}(t) \doteq \frac{1}{2}\,\mathrm{Re} \int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\, d\boldsymbol{q}\, \left|M_{\boldsymbol{kpq}}\right|^2 \bar{\Theta}^*_{\boldsymbol{kpq}}(t),$$

and

$$\bar{\Theta}_{\boldsymbol{kpq}}(t) \doteq \int_{-\infty}^{t} d\bar{t}\, R_{\boldsymbol{k}}(t,\bar{t})\, C_{\boldsymbol{p}}(t,\bar{t})\, C_{\boldsymbol{q}}(t,\bar{t}),$$

given initial conditions at $t = -\infty$. Let us now find determine steady-state self-similar inertial-range solutions in d dimensions to closures of the form(1.L.1). The turbulence could be forced with a linear instability, incorporated with dissipation into the linear coefficient $\nu_{\boldsymbol{k}}$, or else a random force could be added to the right-hand side of Eq. (1.L.1).

By definition, both the external forcing and dissipation $\nu_{\boldsymbol{k}}$ vanish in the inertial range. The symmetry Eq. (1.K) then implies that the nonlinear terms in Eq. (1.L.1), weighted by certain real quantities $\sigma_{\boldsymbol{k}}$, must balance. It is convenient to define

$$\begin{aligned}
&S_{\boldsymbol{k}} \doteq \sigma_{\boldsymbol{k}}(F_{\boldsymbol{k}} - \operatorname{Re} N_{\boldsymbol{k}}) \\
&= \frac{1}{2}\operatorname{Re}\int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\, d\boldsymbol{q}\, \sigma_{\boldsymbol{k}} M_{\boldsymbol{kpq}} M^*_{\boldsymbol{kpq}} \bar{\Theta}_{\boldsymbol{kpq}} + \operatorname{Re}\int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\, d\boldsymbol{q}\, \sigma_{\boldsymbol{k}} M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} \bar{\Theta}^*_{\boldsymbol{pqk}} \\
&= -\frac{1}{2}\operatorname{Re}\int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\, d\boldsymbol{q}\, M_{\boldsymbol{kpq}}(\sigma_{\boldsymbol{p}} M^*_{\boldsymbol{pqk}} + \sigma_{\boldsymbol{q}} M^*_{\boldsymbol{qkp}}) \bar{\Theta}_{\boldsymbol{kpq}} \\
&\quad + \operatorname{Re}\int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\, d\boldsymbol{q}\, \sigma_{\boldsymbol{k}} M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} \bar{\Theta}^*_{\boldsymbol{pqk}} \\
&= -\operatorname{Re}\int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\, d\boldsymbol{q}\, \sigma_{\boldsymbol{p}} M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} \bar{\Theta}_{\boldsymbol{kpq}} + \operatorname{Re}\int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\, d\boldsymbol{q}\, \sigma_{\boldsymbol{k}} M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} \bar{\Theta}^*_{\boldsymbol{pqk}} \\
&= \operatorname{Re}\int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\, d\boldsymbol{q}\, M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} (\sigma_{\boldsymbol{k}} \bar{\Theta}^*_{\boldsymbol{pqk}} - \sigma_{\boldsymbol{p}} \bar{\Theta}_{\boldsymbol{kpq}}). \qquad (1.26)
\end{aligned}$$

Let us seek self-similar solutions of the DIA that obey the scalings (for $\lambda > 0$)

$$M_{\lambda\boldsymbol{k},\lambda\boldsymbol{p},\lambda\boldsymbol{q}} = \lambda^m M_{\boldsymbol{kpq}},$$

$$\sigma_{\lambda\boldsymbol{k}} = \lambda^s \sigma_{\boldsymbol{k}},$$

$$R_{\lambda\boldsymbol{k}}(t,t') = R_{\boldsymbol{k}}(t, t - \lambda^{-\ell}(t-t')),$$

$$C_{\lambda\boldsymbol{k}}(t,t') = \lambda^n C_{\boldsymbol{k}}(t, t - \lambda^{-\ell}(t-t')),$$

so that, upon making the change of variables $\bar{s} \doteq t - \lambda^{-\ell}(t - \bar{t})$ in Eq. (1.L.1),

$$\bar{\Theta}_{\lambda\boldsymbol{k},\lambda\boldsymbol{p},\lambda\boldsymbol{q}} = \lambda^{\ell+2n} \bar{\Theta}_{\boldsymbol{kpq}}.$$

Once we have determined suitable values of the scaling exponent n, we may use Parseval's theorem to compute the wavenumber exponent β for the energy spectrum $E(k) \sim \epsilon^\alpha k^\beta$. If the total energy E is related to the correlation function $C_{\boldsymbol{k}}$ of the fundamental variable ψ by

$$\begin{aligned}
E &= \int d\boldsymbol{k}\, k^\gamma C_{\boldsymbol{k}} \\
&= \int dk\, E(k),
\end{aligned}$$

then $\beta = d - 1 + \gamma + n$.

Following Orszag [1977], we will use the change of variables

$$\begin{aligned}
z &= \frac{k^2}{p}, \qquad w \\
&= \frac{kq}{p}
\end{aligned}$$

to determine values of the exponents ℓ and n for which the angular average $S(k)$ of $S_{\boldsymbol{k}}$ vanishes. In terms of the scaling factor $\lambda = k/z$ we note that

$$\begin{aligned} k &= \lambda z \qquad p \\ &= \lambda k, \qquad q \\ &= \lambda w. \end{aligned}$$

Letting $\boldsymbol{z} = z\hat{\boldsymbol{p}}$ and $\boldsymbol{\omega} = w\hat{\boldsymbol{q}}$, we may then express

$$d\boldsymbol{p}\, d\boldsymbol{q} = \lambda^{3d} d\boldsymbol{z}\, d\boldsymbol{\omega}$$

and

$$\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}) = \lambda^{-d}\delta(z\hat{\boldsymbol{k}} + k\hat{\boldsymbol{p}} + \boldsymbol{\omega}).$$

Hence, upon interchanging $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{k}}$ in the integration, we deduce

$$\begin{aligned} S(k) &\doteq \int d\hat{\boldsymbol{k}}\, S_{\boldsymbol{k}} \\ &= \mathrm{Re} \int d\hat{\boldsymbol{k}} \int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{z}\, d\boldsymbol{\omega}\, \lambda^{3d-d+2m+s+\ell+2n}\, M_{\boldsymbol{z},\boldsymbol{k},\boldsymbol{\omega}} M^*_{\boldsymbol{k},\boldsymbol{\omega},\boldsymbol{z}} (\sigma_{\boldsymbol{z}} \bar{\Theta}^*_{\boldsymbol{k},\boldsymbol{\omega},\boldsymbol{z}} - \sigma_{\boldsymbol{k}} \bar{\Theta}_{\boldsymbol{z},\boldsymbol{k},\boldsymbol{\omega}}) \\ &= -\mathrm{Re} \int d\hat{\boldsymbol{k}} \int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{z}\, d\boldsymbol{\omega}\, \lambda^{2d+2m+s+\ell+2n} M^*_{\boldsymbol{k},\boldsymbol{z},\boldsymbol{\omega}} M_{\boldsymbol{z},\boldsymbol{\omega},\boldsymbol{k}} (\sigma_{\boldsymbol{k}} \bar{\Theta}_{\boldsymbol{z},\boldsymbol{\omega},\boldsymbol{k}} - \sigma_{\boldsymbol{z}} \bar{\Theta}^*_{\boldsymbol{k},\boldsymbol{z},\boldsymbol{\omega}}) \\ &= -S(k), \end{aligned} \tag{1.28}$$

provided that

$$2d + 2m + s + \ell + 2n = 0.$$

The condition(1.L.1) guarantees that the angle-averaged nonlinear terms in Eq. (1.L.1) will balance in a steady state and lead to an inertial range.

The exponent ℓ can be determined by integrating the DIA response function equation

$$\begin{aligned} \frac{\partial}{\partial t} R_{\boldsymbol{k}}(t,t') - \int_{-\infty}^{t} d\bar{t} \int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\, d\boldsymbol{q}\, M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} R^*_{\boldsymbol{p}}(t,\bar{t})\, C^*_{\boldsymbol{q}}(t,\bar{t})\, R_{\boldsymbol{k}}(\bar{t},t') \\ = \delta(t-t'), \end{aligned}$$

over all t', using the steady-state condition

$$\lim_{t\to\infty} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} dt'\, R(t,t') = 0.$$

One obtains

$$-\int_{-\infty}^{\infty} d\bar{t} \int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\, d\boldsymbol{q}\, M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} R^*_{\boldsymbol{p}}(\infty,\bar{t})\, C^*_{\boldsymbol{q}}(\infty,\bar{t}) \int_{-\infty}^{\infty} dt'\, R_{\boldsymbol{k}}(\bar{t},t') = 1.$$

Upon replacing $\boldsymbol{k}$ by $\lambda\boldsymbol{k}$ (for any constant λ) and exploiting the self-similar scalings given in Eqs. (1.27), we make the change of variable $s' = \bar{t} - \lambda^{-\ell}(\bar{t} - t')$ to obtain

$$-\lambda^{d+2m+\ell+n}\int_{-\infty}^{\infty} d\bar{t}\int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\, d\boldsymbol{q}\, M_{\boldsymbol{kpq}}M^*_{\boldsymbol{pqk}}R^*_{\boldsymbol{p}}(\infty,\bar{s})\, C^*_{\boldsymbol{q}}(\infty,\bar{s})\int_{-\infty}^{\infty} ds'\, R_{\boldsymbol{k}}(\bar{t},s') = 1,$$

where $\bar{s} \doteq t-\lambda^{-\ell}(t-\bar{t})$. The integral over $\bar{t}$ is dominated by contributions from large $\bar{t}$, for which the integral over s' asymptotically approaches a constant (with respect to $\bar{t}$), according to Eq. (1.L.1). Hence, after making a final change of variables from $\bar{t}$ to $\bar{s}$, we see that the balance expressed in Eq. (1.L.1) is recovered if

$$\lambda^{d+2m+2\ell+n} = 1,$$

from which we conclude that $\ell = -(d+n)/2 - m$. If one inserts this result into Eq. (1.L.1), one obtains the Kolmogorov scalings

$$\ell = \frac{1}{3}s - \frac{2}{3}m,$$

$$n = -d - \frac{2}{3}(m+s),$$

$$\beta = \gamma - 1 - \frac{2}{3}(m+s).$$

Alternatively, one could adopt instead of Eq. (1.L.1) the stronger condition of *stationarity*, $R_{\boldsymbol{k}}(t,t') = r_{\boldsymbol{k}}(t-t')$ and $C_{\boldsymbol{k}}(t,t') = c_{\boldsymbol{k}}(t-t')$. Equation (1.L.1) is then readily seen to follow directly from Eq. (1.L.1). In either case we have only shown that Eq. (1.L.1) is a necessary condition for self-similar solutions of the form(1.27) to exist. In order that these solutions actually satisfy Eq. (1.L.1), it is also necessary at the very least that the wavenumber integral in Eq. (1.L.1) converges.

Unfortunately, the scaling expressed in Eq. (1.L.1) often leads to a divergence of the q integral in Eq. (1.L.1), preventing self-similar solutions from existing. Typically, the mode-coupling coefficients $M_{\boldsymbol{k},-\boldsymbol{k}-\boldsymbol{q},\boldsymbol{q}}$ asymptotically approach a constant as q goes to zero while $\boldsymbol{k}$ is held fixed. Upon performing the $\boldsymbol{p}$ integration in Eq. (1.L.1), we then see that the q integrand will scale like $q^{d-1}C^*_{\boldsymbol{q}}(t,\bar{t})$ for small q. If $C_{\boldsymbol{q}}$ asymptotically scales as q^n, then the integrand will scale like q^{d-1+n}. But Eq. (1.29) implies that $d-1+n = -1-2(m+s)/3$. Normally $m+s > 0$ (see Table 1.1); in these cases there would be a divergence of the q integral in Eq. (1.L.1) if self-similar solutions really were to exist [Edwards 1964], [Leslie 1973].

This divergence indicates that the dominant contributions to the eddy-turnover time come from the energy spectrum at large scales, where self-similarity no longer holds. (For this reason, the DIA is not invariant to random Galilean transformations.) The actual value of the scaling ℓ that appears in the DIA response must be calculated by taking into account that $C_{\boldsymbol{q}}$ does not actually behave as q^n for small q. The DIA equations apply to the case of zero mean flow, where the energy spectrum goes to zero at low wavenumbers. This means that the integration in Eq. (1.L.1) must be effectively cut off at some fixed large-scale wavenumber k_0. The introduction of this cutoff wavenumber removes the divergence in the integral, but it also changes the above scaling argument. Since the dominant contribution to Eq. (1.L.1) still comes from small q, we need to identify the scaling of the mode-coupling coefficients with k for $q \ll k$,

$$M_{\lambda\boldsymbol{k},-\lambda\boldsymbol{k},\lambda\boldsymbol{q}} = \lambda^{m'} M_{\boldsymbol{k},-\boldsymbol{k},\boldsymbol{q}} \quad (q \ll k).$$

Since the lower wavenumber limit is now fixed, no self-similar scaling in $\boldsymbol{q}$ can be made; the scaling with k for small q then leads to $\lambda^{2\ell+2m'} = 1$. Hence for the DIA equations the actual scalings of the response function, correlation function, and energy spectrum are given by

$$\ell_{\rm DIA} = -m',$$

$$n_{\rm DIA} = -d - m + \frac{m' - s}{2},$$

$$\beta_{\rm DIA} = \gamma - 1 - m + \frac{m' - s}{2}.$$

In Table 1.1 we compare the scalings in Eqs. (1.29) with the anomalous DIA scalings given by Eq. (1.30). The scalings given by Eq. (1.29) are consistent with Kolmogorov's dimensional analysis. We emphasize that these scalings would have also been obtained for the DIA equations (they too are dimensionally consistent) had the wavenumber integral in Eq. (1.L.1) converged.

1.M Test-Field Model (TFM)

The test-field model [Kraichnan 1971b], [Kraichnan 1972] approximates the DIA time-history convolutions in favour of a characteristic "triad interaction time" $\theta_{\boldsymbol{kpq}}$.

Advantages of the TFM:

- Predicts the correct k^{-3} inertial-range energy spectrum.
- Much faster than DIA.

Disadvantages:

- Only predicts equal-time spectral information.
- Does not take account of time-history effects accurately.
- *Assumes* a fluctuation-dissipation relation (not exact except in thermal equilibrium).
- Can predict negative energies if wave effects are present!

Cascade	ψ	d	s	γ	m	m'	ℓ	n	β	$\ell_{\rm DIA}$	$n_{\rm DIA}$	$\beta_{\rm DIA}$
2D enstrophy	ψ	2	4	2	2	1	0	-6	-3	-1	$-\frac{11}{2}$	$-\frac{5}{2}$
2D energy	ψ	2	2	2	2	1	$-\frac{2}{3}$	$-\frac{14}{3}$	$-\frac{5}{3}$	-1	$-\frac{9}{2}$	$-\frac{3}{2}$
3D energy	u	3	0	0	1	1	$-\frac{2}{3}$	$-\frac{11}{3}$	$-\frac{5}{3}$	-1	$-\frac{7}{2}$	$-\frac{3}{2}$
3D helicity	u	3	1	0	1	1	-1	$-\frac{13}{3}$	$-\frac{7}{3}$	-1	-4	-2

Table 1.1: Scaling exponents for two and three-dimensional cascades.

1.N Realizable Test-Field Model (RTFM)

- The realizable test-field model [Bowman and Krommes 1997] adopts a nonstationary form for the Fluctuation Dissipation relation [Bowman *et al.* 1993]. For $t \geq t'$,

$$\underbrace{\overbrace{\frac{C_{\boldsymbol{k}}(t,t')}{C_{\boldsymbol{k}}^{1/2}(t)\,C_{\boldsymbol{k}}^{1/2}(t')}}^{\text{finite amplitude}}}_{\text{correlation coefficient}} = \underbrace{\overbrace{R_{\boldsymbol{k}}(t,t')}^{\text{infinitesimal}}}_{\text{response function}}.$$

*i.e.*Time scales for *amplitude decorrelation* and *decay of infinitesimal disturbances* are comparable since these processes both occur by interaction with the turbulent background.

- The RTFM always predicts non-negative energies.
- The RTFM has an underlying Langevin equation, which, unlike the TFM, does not assume δ-correlated statistics for the noise term f:

$$\frac{\partial}{\partial t}\psi + \eta\psi = f.$$

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