1. Use the phenomenological arguments of Kolmogorov and Kraichnan to determine the exponent of the inertial-range energy spectrum power law consistent with a cascade characterized by $k$-independent helicity transfer.

The helicity transfer is proportional to the quantity

$$\Pi_H(k) = \left[ \int_0^k k^2 E(k) dE \right]^{1/2} k^2 E(k).$$

Let $f(k) = k^2 E(k)$, so that

$$\frac{\Pi}{f^2} = \int_0^k f dk.$$

Differentiate this expression with respect to $k$ to obtain

$$-2\Pi^2 \frac{f'}{f^4} = 1.$$

Let $k_0$ be the smallest wavenumber in the inertial range. Integrate between $k_0$ and $k$ to obtain

$$E(k) = k^{-7/3} \left[ \frac{3}{2} \Pi_H^2 \left( 1 - \frac{k_0}{k} \right) + \left( \frac{k_0}{k} \right) k_0^{-7} E^{-3}(k_0) \right]^{-1/3}, \quad (k \geq k_0).$$

We can rewrite this as

$$E(k) = \left( \frac{2}{3} \right)^{4/3} \Pi_H^{2/3} k^{-7/3} \chi^{-1/3}(k), \quad (k \geq k_0),$$

where

$$\chi(k) = 1 - \frac{k_0}{k} (1 - \chi_0)$$

and $\chi_0 = 2\Pi_H^2 k_0^{-7} E^{-3}(k_0)/3 = \chi(k_0) > 0$. Notice that for $k \gg k_0 |1 - \chi_0|$ we have $E(k) \sim k^{-7/3}$.

2. (a) Show that for two-dimensional unforced incompressible turbulence, the pressure $P$ is related to the stream function $\psi$ by

$$\nabla^2 P = 2 \det \begin{pmatrix} \psi_{xx} & \psi_{xy} \\ \psi_{yx} & \psi_{yy} \end{pmatrix}. $$
The velocity field is $u = \hat{z} \times \nabla \psi = (-\psi_y, \psi_x)$. The Laplacian of the pressure satisfies

\[ \nabla^2 P = -\nabla \cdot (u \cdot \nabla u) = -\frac{\partial}{\partial x}[\psi_y (\psi_{yx}) - \psi_x \psi_{yy}] - \frac{\partial}{\partial y}[-\psi_y (\psi_{xx}) + \psi_x \psi_{xy}] \]

\[ = -\psi_y^2 - \psi_y \psi_{yx} + \psi_{xx} \psi_{yy} + \psi_x \psi_{yy} + \psi_{yy} \psi_{xx} + \psi_y \psi_{xx} - \psi_{xx} \psi_{yy} \]

\[ = 2(\psi_{xx} \psi_{yy} - \psi_{xx}^2) \]

\[ = 2(\psi_{xx} \psi_{yy} - \psi_{yy} \psi_{xx}). \]

(b) What is the pressure field required to maintain the incompressibility of the velocity field $u = (\sin y, \sin x, 0)$?

Let $u = \sin y, v = \sin x$. From part (a), we see that

\[ \nabla^2 P = 2(-v_x u_y - u_x^2) = -2 \cos x \cos y \]

from which we deduce $P(x, y) = \cos x \cos y$ (plus any solution of Laplace's equation that satisfies the boundary conditions; for periodic boundary conditions, this means to within a constant).

3. In two dimensions, the statistical equipartition theory predicts that the ensemble-averaged energy for the inviscid unforced incompressible Navier-Stokes (Euler) equation should be distributed according to

\[ E_k = \frac{1}{2} \left( \frac{1}{\alpha + \beta k^2} \right), \]

where the constants $\alpha$ and $\beta$ are related to the total energy $E = \sum_k E_k$ and enstrophy $Z$ in the flow. Here the sum is over all excited (nonzero) Fourier modes.

(a) Given $\alpha$ and $\beta$, it is straightforward to calculate $E$ and $Z$. What is the formula for $Z$?

\[ Z = \frac{1}{2} \sum_k \left( \frac{k^2}{\alpha + \beta k^2} \right), \]

(b) Given $E$ and $Z$, the inverse problem of determining $\alpha$ and $\beta$ is more difficult. Suppose that only a finite number $2N$ of Fourier modes are excited. Show that the problem of determining $(\alpha, \beta)$ from $(E, Z)$ may be reduced to the problem of solving for the root of a single nonlinear equation.

The constants $\alpha$ and $\beta$ may be determined from the initial energy $E$ and enstrophy $Z$ by expressing the ratio $r = \frac{Z}{E}$ in terms of $\rho = \frac{\alpha}{\beta}$, using the relation

\[ Z = \frac{1}{2\beta} \sum_k \left( 1 - \frac{\alpha}{\alpha + \beta k^2} \right) = \frac{1}{\beta} (N - \alpha E). \]
We find that
\[ r = \frac{N}{\beta E} - \rho, \]
or
\[ r = 2N \left[ \sum_k \frac{1}{\rho + k^2} \right]^{-1} - \rho. \]

Upon inverting the last equation for \( \rho(r) \) with a numerical root solver, we may determine \( \alpha \) and \( \beta \) from the relations
\[ \beta = \frac{N}{\rho E + Z}, \quad \alpha = \rho \beta. \]

(c) Does a solution to (b) exist for all possible combinations of \( E \) and \( Z \)? Why or why not?
No, because if \( k_{\text{min}} \leq |k| \leq k_{\text{max}} \), then
\[ \frac{1}{2} k_{\text{min}}^2 \sum_k \left( \frac{1}{\alpha + \beta k^2} \right) \leq \frac{1}{2} \sum_k \left( \frac{k^2}{\alpha + \beta k^2} \right) \leq \frac{1}{2} k_{\text{max}}^2 \sum_k \left( \frac{1}{\alpha + \beta k^2} \right), \]
so that
\[ k_{\text{min}}^2 E \leq Z \leq k_{\text{max}}^2 E. \]
Therefore if the quantity \( r = Z/E \) lies outside the interval \([k_{\text{min}}^2, k_{\text{max}}^2]\), then no solution can exist.

(d) In three-dimensional inviscid turbulence, one obtains an equipartition of the modal energies \( E_k \), since the Lagrange multiplier \( \beta \) corresponding to the enstrophy is zero. What quantity is in equipartition in two dimensions, when \( \alpha \) and \( \beta \) are both nonzero?
\[ (\alpha + \beta k^2)|u_k|^2 \]

4. Consider two-dimensional flow in a plane perpendicular to \( \mathbf{\hat{z}} \).
(a) Show that the tensor
\[ \epsilon_{kpq} \doteq (\mathbf{\hat{z}} \cdot \mathbf{p} \times \mathbf{q}) \delta_{k+p+q,0} \]
is antisymmetric under interchange of any two indices.
The antisymmetry with respect to interchange of the last two indices follows from the antisymmetry of the cross product. Also,
\[ \epsilon_{pkq} = (\mathbf{\hat{z}} \cdot \mathbf{k} \times \mathbf{q}) \delta_{k+p+q,0} = - (\mathbf{\hat{z}} \cdot \mathbf{p} \times \mathbf{q}) \delta_{k+p+q,0} = -\epsilon_{kpq} \]
and
\[ \epsilon_{qpk} = -\epsilon_{qpk} = \epsilon_{pkq} = -\epsilon_{kpq}. \]
(b) Prove that the two-dimensional Euler equation

\[
\frac{\partial \omega_k}{\partial t} = \frac{\epsilon_{kpq}}{q^2} \omega^*_p \omega^*_q
\]

may be written as the noncanonical Hamiltonian system

\[
\dot{\omega}_k = J_{kq} \frac{\partial H}{\partial \omega_q},
\]

(where \( H \) is the Hamiltonian, in this case the total energy), by showing that the symplectic tensor

\[
J_{kq} = \epsilon_{kpq} \omega^*_p
\]

obeys both the antisymmetry

\[
J_{kq} = -J_{qk}
\]

and the Jacobi identity

\[
J_{k\ell} \frac{\partial J_{pq}}{\partial \omega_\ell} + J_{p\ell} \frac{\partial J_{qk}}{\partial \omega_\ell} + J_{q\ell} \frac{\partial J_{kp}}{\partial \omega_\ell} = 0.
\]

The Euler equation can be put into the above Hamiltonian form since

\[
H = \frac{1}{2} \sum_k \frac{1}{k^2} |\omega_k|^2 = \frac{1}{2} \sum_k \frac{\omega_k \omega_{-k}}{k^2} \Rightarrow \frac{\partial H}{\partial \omega_q} = \frac{1}{2} \omega^*_q + \frac{1}{2} \omega_{-q} = \frac{\omega^*_q}{q^2}.
\]

To prove that the result is indeed a Hamiltonian, we first note that \( J_{kq} \) inherits the antisymmetry of \( \epsilon_{kpq} \) that we established in part (a). To establish the Jacobi symmetry, we first compute

\[
J_{k\ell} \frac{\partial J_{pq}}{\partial \omega_\ell} = \epsilon_{kj\ell} \epsilon_{jpr} q \frac{\partial \omega_{-r}}{\partial \omega_\ell} = \epsilon_{kj\ell} \epsilon_{p(-\ell)q} \omega^*_j = \hat{z} \cdot k \times (p + q) \hat{z} \cdot p \times q \delta(k + j + p + q) \omega^*_j
\]

Since \( \epsilon_{kpq} \) is invariant under cyclic permutations of its indices, the sum of the cyclic permutations of \( J_{k\ell} \frac{\partial J_{pq}}{\partial \omega_\ell} \) may be written as \( A_{kpq} \omega_{k+p+q} \), where

\[
A_{kpq} = \hat{z} \cdot k \times (p + q) \hat{z} \cdot p \times q + \hat{z} \cdot p \times (q + k) \hat{z} \cdot q \times k + \hat{z} \cdot q \times (k + p) \hat{z} \cdot k \times p.
\]

Of the six terms in the above expression, the first and last, the second and third, and the fourth and fifth cancel each other pairwise, so that \( A_{kpq} = 0 \).
5. (a) Prove the Gaussian integration by parts formula

\[ \langle vf(v) \rangle = \langle v^2 \rangle \left\langle \frac{\partial f}{\partial v} \right\rangle \]

for a (scalar) centered Gaussian random variable \( v \) and a continuously differentiable \((C^1)\) function \( f : \mathbb{R} \rightarrow \mathbb{R} \) that vanishes at \( \pm \infty \).

\[ \langle vf(v) \rangle = \int vf(v) dP = \frac{1}{\sqrt{2\pi\sigma}} \int f(v)ve^{-\frac{v^2}{2\sigma^2}} dv \]

\[ = -\frac{\sigma^2}{\sqrt{2\pi\sigma}} \int f(v) \frac{\partial}{\partial v} e^{-\frac{v^2}{2\sigma^2}} dv \]

\[ = \frac{\sigma^2}{\sqrt{2\pi\sigma}} \int \frac{\partial f}{\partial v} e^{-\frac{v^2}{2\sigma^2}} dv \]

\[ = \sigma^2 \int \frac{\partial f}{\partial v} dP = \sigma^2 \left\langle \frac{\partial f}{\partial v} \right\rangle. \]

Upon setting \( f(v) = v \) we see that the second moment of \( v \) is just the variance of \( v \):

\[ \langle v^2 \rangle = \sigma^2 \langle 1 \rangle = \sigma^2. \]

Hence

\[ \langle vf(v) \rangle = \langle v^2 \rangle \left\langle \frac{\partial f}{\partial v} \right\rangle. \]

(b) Use part (a) to show that that the odd-order moments of a centered Gaussian distribution are zero.

First, we note that

\[ \langle v \rangle = 0 \]

since \( v \) is centered.

Part (a) implies that

\[ \langle v^{2n+1} \rangle = \langle v^2 \rangle \left\langle \frac{\partial v^{2n}}{\partial v} \right\rangle = 2n \langle v^2 \rangle \langle v^{2n-1} \rangle. \]

Therefore, by induction, \( \langle v^{2n-1} \rangle = 0 \) for all \( n \in \mathbb{N} \).