Chapter 3
Abstract Measure Spaces

Let $X$ be a set. A $\sigma$-algebra on $X$ is a collection $\mathcal{B}$ of $X$ such that

(i) $\emptyset \in \mathcal{B}$;  
(ii) If $S \in \mathcal{B}$, then the complement $S^c = X \setminus S$ is also an element of $\mathcal{B}$;  
(iii) If $S_1, S_2, \ldots \in \mathcal{B}$, then $\bigcup_{n=1}^{\infty} S_n \in \mathcal{B}$.

Let $\mathcal{B}$ be a $\sigma$-algebra on a set $X$. A measure $\mu$ on $\mathcal{B}$ is a map $\mu : \mathcal{B} \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$;  
(ii) If $S_1, S_2, \ldots$ are disjoint elements of $\mathcal{B}$, then $\mu(\bigcup_{k=1}^{\infty} S_k) = \sum_{k=1}^{\infty} \mu(S_k)$.

Let $(X, \mathcal{B}, \mu)$ be a measure space.

(i) If $S_1, S_2, \ldots$ are $\mathcal{B}$-measurable, then

$$\mu\left(\bigcup_{k=1}^{\infty} S_k\right) \leq \sum_{k=1}^{\infty} \mu(S_k).$$

(ii) If $S_1 \subset S_2 \subset \ldots$ is an increasing sequence of $\mathcal{B}$-measurable sets, then

$$\mu\left(\bigcup_{k=1}^{\infty} S_k\right) = \lim_{n \to \infty} \mu(S_n) = \sup_{n} \mu(S_n).$$
(iii) If \( S_1 \supset S_2 \supset \ldots \) is a decreasing sequence of \( B \)-measurable sets and at least one of the \( \mu(S_k) \) is finite, then
\[
\mu\left( \bigcap_{k=1}^{\infty} S_k \right) = \lim_{n \to \infty} \mu(S_n) = \inf_n \mu(S_n).
\]
downward monotone convergence

- Let \((X, \mathcal{B}, \mu)\) be a measure space. Suppose \( S_n, n = 1, 2, \ldots \) are \( \mathcal{B} \)-measurable sets that converge to a set \( S \). Then
  (i) \( S \) is \( \mathcal{B} \)-measurable.
  (ii) If the \( S_n \) are all contained in another \( \mathcal{B} \)-measurable set of finite measure, then \( m(S_n) \) converges to \( m(S) \).

Problem 3.8: (Characterization of measurable functions)

Let \((X, \mathcal{B})\) be a measurable space. Show that

(i) a function \( f : X \to [0, \infty] \) is measurable iff the level sets \( \{x \in X : f(x) > \lambda\} \) are measurable for every \( \lambda \in [0, \infty) \);

(ii) an indicator function \( 1_S \) of a set \( S \subset X \) is measurable iff \( S \) is measurable;

(iii) a function \( f : X \to [0, \infty] \) (or \( f : X \to \mathbb{C} \)) is measurable iff \( f^{-1}(S) \) is measurable for every Borel-measurable subset \( S \) of \([0, \infty] \) (or \( \mathbb{C} \));

(iv) a function \( f : X \to \mathbb{C} \) is measurable iff its real and imaginary parts are measurable;

(v) a function \( f : X \to \mathbb{R} \) is measurable iff its positive and negative parts are measurable;

(vi) the pointwise limit \( f \) of a sequence of measurable functions \( f_n : X \to [0, \infty] \) (or \( \mathbb{C} \)) is also measurable;

(vii) if \( f : X \to [0, \infty] \) (or \( \mathbb{C} \)) is measurable and \( \phi : [0, \infty] \to [0, \infty] \) (or \( \mathbb{C} \to \mathbb{C} \)) is continuous, then \( \phi \circ f \) is measurable;

(viii) the sum or product of two measurable functions in \([0, \infty] \) (or \( \mathbb{C} \)) is measurable.

Theorem 3.7 (Monotone convergence theorem): Let \((X, \mathcal{B}, \mu)\) be a measure space and \( f_1 \leq f_2 \leq \ldots \) be an increasing sequence of unsigned measurable functions on \( X \). Then
\[
\int_X \lim_{n \to \infty} f_n \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu.
\]
Corollary 3.7.3 (Fatou’s lemma): Let \((X, \mathcal{B}, \mu)\) be a measure space and \(f_1, f_2, \ldots\) be a sequence of unsigned measurable functions on \(X\). Then
\[
\int_X \liminf_{n \to \infty} f_n \, d\mu \leq \liminf_{n \to \infty} \int_X f_n \, d\mu.
\]

Theorem 3.8 (Dominated convergence theorem): Let \((X, \mathcal{B}, \mu)\) be a measure space and \(f_1, f_2, \ldots\) be a sequence of complex-valued measurable functions on \(X\) that converge pointwise \(\mu\)-almost everywhere on \(X\). Suppose that there exists an unsigned absolutely integrable function \(G : X \to [0, \infty]\) such that for each \(n \in \mathbb{N}\), \(|f_n| \leq G\) \(\mu\)-almost everywhere. Then
\[
\int_X \lim_{n \to \infty} f_n \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu.
\]
Chapter 4

Modes of Convergence

• pointwise almost everywhere
• uniformly almost everywhere (in $L^\infty$ norm)
• almost uniformly
• in $L^1$ norm
• in measure
Chapter 5

Differentiation Theorems

Theorem 5.1 (Lebesgue differentiation theorem on $\mathbb{R}$): Let $f : \mathbb{R} \to \mathbb{C}$ be an absolutely integrable function. Then
\[
\lim_{h \to 0^+} \frac{1}{h} \int_{[x, x+h]} f(t) \, dt = f(x)
\]
and
\[
\lim_{h \to 0^+} \frac{1}{h} \int_{[x-h, x]} f(t) \, dt = f(x)
\]
for almost every $x \in \mathbb{R}$.

Theorem 5.3 (Monotone differentiation theorem): Every monotone function $f : \mathbb{R} \to \mathbb{R}$ is differentiable almost everywhere.

Definition: The total variation of a function $F : \mathbb{R} \to \mathbb{R}$ on an (finite or infinite) interval $I$ is
\[
|F|_{TV(I)} = \sup_{x_0 < \ldots < x_n} \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})|.
\]
If $|F|_{TV(I)}$ is finite, we say that $F$ has bounded variation on $I$. If $F$ has bounded variation on $\mathbb{R}$, we say that $F$ has bounded variation.

Theorem 5.4: A function $F : \mathbb{R} \to \mathbb{R}$ has bounded variation iff it is the difference of two bounded monotone functions.

Theorem 5.5 (1D Lipschitz differentiation theorem): Every Lipschitz continuous function is locally of bounded variation, and hence differentiable almost everywhere. Furthermore, its derivative, when it exists, is bounded by its Lipschitz constant.

Theorem 5.6 (Upper bound for fundamental theorem): Let $F : [a, b] \to \mathbb{R}$ be increasing, so that the unsigned function $F' : [a, b] \to [0, \infty]$ exists almost everywhere and is measurable. Then
\[
\int_{[a, b]} F' \leq F(b) - F(a).
\]
Definition: A function $F : \mathbb{R} \to \mathbb{R}$ is said to be absolutely continuous if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\sum_{k=1}^{n}|F(b_k) - F(a_k)| < \epsilon$ for every finite collection of disjoint intervals $(a_1, b_1) \ldots (a_n, b_n)$ of total length $\sum_{k=1}^{n}(b_k - a_k) < \delta$.

Theorem 5.7 (Fundamental theorem for absolutely continuous functions): Let $F : [a, b] \to \mathbb{R}$ be absolutely continuous. Then

$$\int_{[a, b]} F' = F(b) - F(a).$$
Chapter 6

Outer Measures, Pre-measures, and Product Measures

**Definition:** Given a set $X$, an outer measure is a map $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that

1. $\mu^*(\emptyset) = 0$; **nullity**
2. $S \subset T \subset X \Rightarrow \mu^*(S) \leq \mu^*(T)$; **monotonicity**
3. $\mu^*\left(\bigcup_{k=1}^{\infty} S_k\right) \leq \sum_{k=1}^{\infty} \mu^*(S_k)$, where $S_k \subset X$. **countable subadditivity**

**Definition:** Let $\mu^*$ be an outer measure on a set $X$. A set $S \subset X$ is said to be Carathéodory measurable if the Carathéodory criterion

$$\mu^*(A) = \mu^*(A \cap S) + \mu^*(A \cap S^c)$$

holds for every set $A \subset X$.

**Markov’s inequality:** For every $\lambda \in (0, \infty)$,

$$\mu\{x \in X : f(x) \geq \lambda\} \leq \frac{1}{\lambda} \int_X f \, d\mu.$$ 

**Theorem 6.1** (Carathéodory lemma): Let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be an outer measure on a set $X$, let $\mathcal{B}$ be the collection of all subsets of $X$ that are Carathéodory measurable with respect to $\mu^*$ and let $\mu : \mathcal{B} \to [0, \infty]$ be the restriction of $\mu^*$ to $\mathcal{B}$. Then $\mathcal{B}$ is a $\sigma$-algebra and $\mu$ is a measure.
Definition: A premeasure on a Boolean algebra $\mathcal{B}_0$ is a finitely additive measure $\mu_0 : \mathcal{B}_0 \to [0, \infty]$ such that $\mu_0(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu_0(E_k)$ whenever $E_1, E_2, \ldots$ are disjoint subsets of $\mathcal{B}_0$ such that $\bigcup_{k=1}^{\infty} E_k \in \mathcal{B}_0$.

Theorem 6.2 (Hahn–Kolmogorov): Every premeasure $\mu_0 : \mathcal{B}_0 \to [0, \infty]$ on a Boolean algebra $\mathcal{B}_0$ in $X$ can be extended to a countably additive measure $\mu : \mathcal{B} \to [0, \infty]$.

Corollary 6.5.3 (Fubini’s theorem): Let $(X, \mathcal{B}_X, \mu_X)$ and $(Y, \mathcal{B}_Y, \mu_Y)$ be complete $\sigma$-finite measure spaces and let $f : X \times Y \to \mathbb{C}$ be absolutely integrable with respect to $\mathcal{B}_X \times \mathcal{B}_Y$. Then

$$\int_{X \times Y} f(x,y) \, d\mu_X \times \mu_Y(x,y) = \int_X \int_Y f(x,y) \, d\mu_Y(y) \, d\mu_X(x) = \int_Y \int_X f(x,y) \, d\mu_X(x) \, d\mu_Y(y).$$