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• Scrap paper is supplied.

• No notes or books are permitted.

• All electronic equipment, including calculators, is prohibited. Make certain that cell phones are turned off. Check that you have 3 pages.

• This exam consists of 3 questions, for a total of 12 points.

If anything is unclear, please ask!
1. If $S \subset \mathbb{R}^d$ is an elementary set, prove directly from the definition of the Jordan inner and outer measures that $m_{*J}(S)$ and $m^{*J}(S)$ equal the Lebesgue measure $m(S)$.

Since $S$ is one of the elementary sets that we need to consider in the supremum,

$$m_{*J}(S) \doteq \sup_{E \subseteq S} m(E) \geq m(S).$$

Also

$$m^{*J}(S) \doteq \inf_{E \supseteq S} m(E) \leq m(S),$$

so that $m_{*J}(S) \geq m^{*J}(S)$. By monotonicity of the elementary measure, we also see that $m_{*J}(S) \leq m^{*J}(S)$. Hence $m_{*J}(S) = m^{*J}(S) = m(S)$.

2. Let $\{B_k\}_{k=1}^{\infty}$ be a sequence of almost disjoint boxes. Prove that the Lebesgue measure of $\bigcup_{k=1}^{\infty} B_k$ is equal to its Jordan inner measure.

By countable additivity,

$$m(S) = \sum_{k=1}^{\infty} |B_k|.$$

For each $n \in \mathbb{N}$, let $T_n = \bigcup_{k=1}^{n} B_k$. Since $T_n \subseteq S$, we know from the definition of Jordan inner measure that

$$m_{*J}(S) \geq m(T_n).$$

From monotone upward convergence, we then see that

$$m_{*J}(S) \geq \lim_{n \to \infty} m(T_n) = m\left(\bigcup_{n=1}^{\infty} T_n\right) = m(S).$$

But we have also seen that

$$m_{*J}(S) \leq m(S).$$

Hence

$$m_{*J}(S) = m(S).$$

3. Families $\mathcal{F}_1$ and $\mathcal{F}_2$ generate the same $\sigma$-algebra if $\langle \mathcal{F}_1 \rangle \supset \mathcal{F}_2$ and $\langle \mathcal{F}_2 \rangle \supset \mathcal{F}_1$. Show that the collection of subsets of $\mathbb{R}^d$ that are:

- open;
- closed;
- compact;
- open balls;
- boxes;
all generate the Borel $\sigma$-algebra on $\mathbb{R}^d$. Use the notation $\langle$open$, \rangle$, $\langle$closed$\rangle$, etc.

- By definition, the Borel $\sigma$-algebra is the $\sigma$-algebra generated by the collection of open sets.
- Since a $\sigma$-algebra is closed under complements, $\langle$open$\rangle$ contains $\langle$closed$\rangle$ and $\langle$closed$\rangle$ contains $\langle$open$\rangle$.
- Since every compact set in $\mathbb{R}^d$ is closed, $\langle$closed$\rangle$ contains $\langle$compact$\rangle$.
- Every open set can be written as a countable union of open balls, so $\langle$open balls$\rangle$ contains $\langle$open$\rangle$. Since open balls are open, $\langle$open$\rangle$ contains $\langle$open balls$\rangle$.
- Since every open set can be written as a countable union of almost disjoint closed cubes, $\langle$boxes$\rangle$ contains $\langle$open$\rangle$. Since closed cubes are compact, $\langle$compact$\rangle$ contains $\langle$open$\rangle$ and hence also $\langle$boxes$\rangle$ (finite intersections of open and closed sets).
- Every elementary set is a finite union of boxes, so $\langle$boxes$\rangle$ contains $\langle$elementary$\rangle$. Since boxes are elementary, $\langle$elementary$\rangle$ contains $\langle$boxes$\rangle$.

There are many other possible routes to showing these equivalent representations of the Borel $\sigma$-algebra. For example, one can show directly that $\langle$compact$\rangle$ contains $\langle$closed$\rangle$ since every closed set $F$ can written as a countable union of compact sets $F \cap B_k[0]$. 