MATH 417 Section Q1

Midterm Exam

Dr. J. Bowman
28 February 2020
10:00–10:50

Name (Last, First): ____________________________

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• Scrap paper is supplied.

• No notes or books are permitted.

• All electronic equipment, including calculators, is prohibited. Make certain that cell phones are turned off. Check that you have 3 pages.

• This exam consists of 3 questions, for a total of 14 points.

If anything is unclear, please ask!
1. For each \( n \in \mathbb{N} \), consider the unsigned functions \( f_n : \mathbb{R} \to [0, \infty) \) defined by

\[
    f_n(x) = \begin{cases} 
        1 - n|x| & \text{if } |x| \leq \frac{1}{n}, \\
        0 & \text{otherwise}.
    \end{cases}
\]

(a) Is each \( f_n \) a simple function? Circle the correct answer: Yes / No.

Each \( f_n \) takes on more than a finite number of values.

(b) Determine the pointwise limit \( f(x) = \lim_{n \to \infty} f_n(x) \).

\[
    f(x) = \begin{cases} 
        1 & \text{if } x = 0, \\
        0 & \text{otherwise}.
    \end{cases}
\]

(c) Is \( f \) continuous? Circle the correct answer: Yes / No.

The function \( f \) takes on values of both 0 and 1 in any neighbourhood of \( x = 0 \).

(d) Is \( f \) measurable? Circle the correct answer: Yes / No.

In fact \( f \) is a simple function.

(e) Is \( f \) a simple function? Circle the correct answer: Yes / No.

The sets \( \{0\} \) and \( \mathbb{R} \setminus \{0\} \) are both Lebesgue measurable.

(f) Evaluate \( \int_{\mathbb{R}} f \).

\[
    \int_{\mathbb{R}} f = 1 \cdot m(\{0\}) + 0 \cdot m(\mathbb{R} \setminus \{0\}) = 1 \cdot 0 + 0 \cdot \infty = 0.
\]

2. Let \( S \) be a subset of \( \mathbb{R}^d \) with finite outer Lebesgue measure \( m^*(S) \).

(a) For each \( j \in \mathbb{N} \), show that there exists a Lebesgue measurable set \( S_j \supset S \) such that \( m(S_j) < m^*(S) + 1/j \).

By the definition of the Lebesgue outer measure, for each \( j \in \mathbb{N} \), the set \( S \) is contained in a countable union \( S_j = \bigcup_{k=1}^{\infty} B_{k,j} \) of boxes \( B_{k,j} \) such that

\[
    \sum_{k=1}^{\infty} |B_{k,j}| < m^*(S) + \frac{1}{j}.
\]

(b) Use your sets \( S_j \) to construct Lebesgue measurable sets \( T_n \supset S \) for each \( n \in \mathbb{N} \) such that \( T_{n+1} \subset T_n \subset S_n \).

The sets \( T_n = \bigcap_{j=1}^{n} S_j \) contains \( S \) and satisfies \( T_{n+1} \subset T_n \subset S_n \).
(c) Show that $S$ is contained in a Lebesgue measurable set $T$, with $m(T) = m^*(S)$.

From monotonicity, we see that

$$m^*(S) \leq m(T_n) \leq m(S_n) < m^*(S) + \frac{1}{n}.$$ 

Let $T = \bigcap_{n=1}^{\infty} T_n$. Since $m(T_1) < m^*(S) + 1 < \infty$, we know from downward monotone convergence that

$$m^*(S) \leq m(T) = \lim_{n \to \infty} m(T_n) \leq m^*(S).$$

Thus $T$ is a Lebesgue measurable set with $m(T) = m^*(S)$.

**Alternative solution:** Since $S \subset T = \bigcap_{n=1}^{\infty} T_n \subset T_j \subset S_j$ for every $j \in \mathbb{N}$ we know from monotonicity that $m^*(S) \leq m(T) \leq m^*(S_j) < m^*(S) + 1/j$ for every $j \in \mathbb{N}$. On taking the limit $j \to \infty$, we find $m(T) = m^*(S)$.

(d) Show that every arbitrary subset $A$ of $\mathbb{R}^d$ is contained in a Lebesgue measurable set $T$ with measure $m(T) = m^*(A)$.

The case were $m^*(A)$ is finite is proven in (c). When $m^*(A) = \infty$, consider $T = \mathbb{R}^d$.

3. Suppose that $S \subset \mathbb{R}^d$ is a Lebesgue null set and let $T \subset S$. Prove that $T$ is Lebesgue measurable and determine $m(T)$.

We are given that $m(S) = 0$. Given $\epsilon > 0$, there exists an open set $U \supset S \supset T$ such that

$$m^*(U \setminus S) < \epsilon.$$ 

Since $U \setminus T \subset (U \setminus S) \cup (S \setminus T)$, we see from subadditivity and monotonicity that

$$m^*(U \setminus T) \leq m^*(U \setminus S) \cup m^*(S \setminus T) < \epsilon + m(S) < \epsilon.$$ 

Thus $T$, is also Lebesgue measurable. It follows from monotonicity that $T$ has Lebesgue measure zero; that is, it also a Lebesgue null set.

**Alternative solution:** If $S$ is a null set, we know from outer regularity that

$$0 \leq m^*(T) = \inf_{U \supseteq T, U \text{ open}} m^*(U) \leq \inf_{U \supseteq S \supseteq T, U \text{ open}} m^*(U) = m^*(S) = 0$$

and we have seen that null sets are Lebesgue measurable, so $m(T) = m^*(T) = 0$.

**Alternative solution:** From monotonicity we know that $0 \leq m^*(T) \leq m(S) = 0$. Hence $T$ is also a null set. We have seen that null sets are Lebesgue measurable, so $m(T) = m^*(T) = 0$. 
