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1. A The problem of measure

The measure $m(E)$ of a solid body $E$ is a fundamental concept in Euclidean geometry. In one, two, and three dimensions, we refer to this measure as the length, area, or volume of $E$ respectively.

In classical geometry, the measure of a body is typically determined by partitioning the body into components that can be translated or rotated and then reassembled into a simpler body with the same measure. Alternatively, lower and upper bounds on the measure of a body can be obtained by computing the measure of some inscribed or circumscribed body. Such arguments were justified by viewing the measure of a macroscopic body as the sum of the measures of its microscopic components.

With the advent of analytic geometry, Euclidean geometry was reinterpreted as the study of Cartesian products $\mathbb{R}^d$ of the real line $\mathbb{R}$. Within this analytical framework, it is no longer intuitively obvious how to define the measure $m(E)$ of a general subset $E$ of $\mathbb{R}^d$. This is known as the problem of measure.

Remark: If one tries to formalize the physical notion of the measure of a body as the sum of the measure of its component “atoms”, one runs into an immediate problem: a typical solid body consists of an uncountably infinite number of points, each of which has zero measure, and the product $\infty \cdot 0$ is indeterminate. Moreover, two bodies with the same number of points need not have the same measure: in one dimension, the intervals $[0, 1]$ and $[0, 2]$ have the same cardinality (using the bijection $x \mapsto 2x$) but different lengths. One can disassemble $[0, 1]$ into an uncountably infinite number of points and reassemble them to form a set of twice the length! Pathological problems of this nature can even occur when one restricts the assembly to a finite number of components. In three or more dimensions, the famous Banach–Tarski paradox illustrates that the unit ball $B$ can be disassembled...
into five pieces that can be translated, rotated, and then reassembled to form two disjoint copies of \( B \! \! \! \)!

**Remark:** The construction of the pathological pieces in the Banach–Tarski paradox requires the controversial *axiom of choice*. While trivial for finite sets (using induction), the axiom of choice for infinite (even countably infinite) sets does not follow from the other axioms of set theory and must be explicitly added to our fundamental list of axioms:

**Axiom 1.1** (Axiom of choice): Let \( \{ E_\alpha : \alpha \in A \} \) be a family of nonempty sets \( E_\alpha \), indexed by elements of \( A \). Then one can construct a set \( \{ x_\alpha : \alpha \in A \} \) of elements \( x_\alpha \) chosen from \( E_\alpha \).

- When \( A = \mathbb{N} \), the axiom of choice states that it is possible to select a sequence \( x_1, x_2, \ldots \) of elements from a sequence of nonempty sets \( E_1, E_2, \ldots \).

Two important definitions of measure are the *Jordan measure*, which underlies the *Riemann integral*. This definition suffices for many branches of mathematics, for example, for defining the area under the graph of a continuous function. However, the notion of Jordan measure turns out to be inadequate for certain sets that arise as limits of other sets. The notion of *Lebesgue measure* and the associated *Lebesgue integral* were developed by the French mathematician Henri Lebesgue in 1902 to fill this gap.

**Definition:** The *indicator function* \( 1_S \) for a set \( S \) is

\[
1_S(x) = \begin{cases} 
1 & \text{if } x \in S \\
0 & \text{otherwise}
\end{cases}
\]

**Remark:** To see why we might need a new type of integral, let \( \{ q_1, q_2, \ldots \} \) be an enumeration of \( \mathbb{Q} \cap [0, 1] \) and for \( n \in \mathbb{N} \) define \( f_n = 1_{q_1,q_2,\ldots,q_n} \). Notice that the sequence of functions \( \{ f_n \} \) converges pointwise on \([0, 1]\) to the *Dirichlet function* \( 1_{\mathbb{Q}\cap[0,1]} \). However, although the functions \( f_1, f_2, \ldots \) are Riemann integrable on \([0, 1]\), with \( \int_0^1 f_n = 0 \), their pointwise limit \( 1_{\mathbb{Q}\cap[0,1]} \) is not, since every nondegenerate interval contains both rational and irrational numbers. Although the interchange of limits and Riemann integration would be guaranteed by uniform convergence, the convergence of \( f_n \) is not uniform on \([0, 1]\).

**Remark:** Similarly, given

\[
f(x) = \int_a^b F(x,t) \, dt,
\]

the statement

\[
f'(x) = \int_a^b \frac{\partial F(x,t)}{\partial x} \, dt
\]

does not hold in general (although it does if \( \partial F(x,t)/\partial x \) is continuous in both \( x \) and \( t \)).
Q. Is it possible to find a new type of integral that generalizes the Riemann integral (so that every Riemann integral is still integrable in the new sense, with the same value) but for which limit processes (limits and integrals, derivatives and integrals) can always be interchanged?

A. No, the following example shows that some restrictions will still be required:

$$f_n(x) = \begin{cases} 
    n^2 x, & \text{if } x \in [0, \frac{1}{2^n}] \\
    n^2 \left( \frac{2}{n} - x \right), & \text{if } x \in \left( \frac{1}{n}, \frac{2}{n} \right] \\
    0 & \text{otherwise.}
\end{cases}$$

We see that \( \lim_{n \to \infty} f_n(x) = 0 \) for \( x \in [0, 1] \) but

$$\int_0^1 \lim_{n \to \infty} f_n = 0 \neq 1 = \lim_{n \to \infty} \int_0^1 f_n.$$

Nevertheless, the class of functions for which such interchange of limit process is valid will be much larger with the Lebesgue integral we are about to develop.

Remark: To get an idea of the difference between the Riemann and Lebesgue integrals, we can think of the Riemann integral as “a zebra with vertical stripes.” For example, we can approximate the Riemann integral of a continuous function by the upper sum

$$\int_a^b f \approx \sum_{k=1}^n f(\xi_k) m(I_k),$$

obtained by partitioning \([a, b]\) into subintervals \(I_k = [x_{k-1}, x_k]\) for \(k = 1, \ldots, n\), and finding points \(\xi_k \in I_k\) at which \(f\) achieves its maximum on \(I_k\), as illustrated below:
In contrast, one can think of the Lebesgue integral as “a zebra with horizontal stripes,” in the sense that one first partitions the $y$ axis into intervals $[y_{k-1}, y_k]$ and then computes the preimages

$$J_k = \{ x \in [a, b] : f(x) \in [y_{k-1}, y_k] \}.$$ 

For the Lebesgue approach, the integral is then approximated as

$$\int_a^b f \approx \sum_{k=1}^n y_k m(J_k),$$

as shown in the following diagram

Q. In the above example, we notice that $J_2$ and $J_3$ are not intervals. What is the “length” of a general subset of $\mathbb{R}$ that is not an interval?

- For example, what is the “length” of the rational numbers within the unit interval $[0, 1]$. This is equivalent to determining the integral

$$\int_0^1 1_{\mathbb{Q}},$$
which, as we have already discussed, is not Riemann integrable. However we will soon see that the Lebesgue integral of \(1_Q\) does exist and evaluates to 0, as one might expect from Cantor’s diagonalization argument (there are vastly more irrationals than rationals within the unit interval). In fact, the Lebesgue measure will be constructed so that the measure of any countable set is zero, precisely so that one can then integrate functions like \(1_Q\) that have only a countable number of discontinuities (if the subsets \(J_k\) that contain the discontinuities have measure zero, they will not contribute to the integral).

Remark: To attempt to answer the above question, we need to make the notion of “length”, or in general, measure, more precise.

Definition: Let \(S\) be a set. We define the power set \(\mathcal{P}(S)\) as the set of all subsets of \(S\):

\[
\mathcal{P}(S) = \{s : s \subset S\}.
\]

- \(\mathcal{P}(\emptyset) = \{\emptyset\}\).
- \(\mathcal{P}\big(\{a, b, c\}\big) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}\).

Problem 1.1: If \(S\) is a finite set of \(n\) elements, use induction to show that the \(\mathcal{P}(S)\) has \(2^n\) elements.

Definition: We define the non-negative extended real numbers \([0, \infty]\), with the convention that \(\infty \cdot 0 = 0 \cdot \infty = 0\).

If we wish to generalize our notion of measure from classical geometry to arbitrary subsets of \(\mathbb{R}^d\), it is reasonable to seek a function \(m(\mathcal{P}(\mathbb{R}^d)) \to [0, \infty]\) that satisfies the following properties:

1. \(m(\emptyset) = 0\); nullity
2. \(m([a_1, b_1] \times \ldots \times [a_d, b_d]) = \prod_{k=1}^{d} (b_k - a_k)\); measure of a box
3. \(m(S + x) = m(S) \quad \forall x \in \mathbb{R}^d, S \subset \mathbb{R}^d\); translational invariance
4. \(m\left(\bigcup_{k=1}^{\infty} S_k\right) = \sum_{k=1}^{\infty} m(S_k)\) for disjoint subsets \(S_k\) of \(\mathbb{R}^d\); countable disjoint additivity
Remark: The above properties imply the \textit{monotonicity} property

\[
S \subset T \Rightarrow m(S) \leq m(T).
\]

To see this set \( S_1 = S, \ S_2 = T \setminus S, \) and \( S_k = \emptyset \) for \( k \geq 3. \) Then

\[
m(S) \leq m(S) + m(T \setminus S) = \sum_{k=1}^{\infty} m(S_k) = m\left( \bigcup_{k=1}^{\infty} S_k \right) = m(T).
\]

Remark: Unfortunately, the following counterexample shows that even in one dimension \((d = 1),\) no such measure \( m \) exists.

- Define an equivalence relation \( \sim \) on \([0,1]:\)

\[
x \sim y \iff x - y \in \mathbb{Q}.
\]

For \( x \in [0,1], \) let \([x] = \{a \in [0,1] : a \sim x\}.\)

Claim: If \([x] \neq [y],\) then \([x] \cap [y] = \emptyset.\)

Proof: Given \( a \in [x] \) and \( b \in [x] \cap [y],\) we see that \( a \sim b \) and also \( b \sim y,\) which implies that \( a \in [y].\) Thus \([x] \subset [y].\) Similarly, \([y] \subset [x].\) \hfill \ensuremath{\square}

Construct a set \( S \) such that for each \( x \in [0,1], \) \( S \cap [x] \) consists of precisely one element (this requires the axiom of choice).

Claim:

\[
[0,1] \subset \bigcup_{q \in \mathbb{Q} \cap [-1,1]} q + S \subset [-1,2].
\]

Proof: The second inclusion is clear. To see the first, let \( x \in [0,1]. \) Then there exists \( y \in S \ni y \in [x] \) so that \( q = x - y \in \mathbb{Q} \cap [-1,1]. \) Hence \( x \in q + S. \) \hfill \ensuremath{\square}

Let \( q_1, q_2, \ldots \) be an enumeration of \( \mathbb{Q} \cap [-1,1] \) such that \( q_n \neq q_m \) for \( n \neq m.\)

Claim: \((q_n + S) \cap (q_m + S) = \emptyset \) for \( n \neq m.\)

Proof: Let \( x \in (q_n + S) \cap (q_m + S). \) Then \( \exists s_n, s_m \in S \ni x = q_n + s_n = q_m + s_m. \)

Hence \([s_n] = [x] = [s_m],\) which, by the choice of \( S,\) implies that \( s_n = s_m.\) Thus \( q_n = q_m \) and hence \( n = m. \) If \( m(S) = 0, \) then

\[
1 = m([0,1]) \leq m\left( \bigcup_{k=1}^{\infty} q_k + S \right) = \sum_{k=1}^{\infty} m(q_k + S) = \sum_{k=1}^{\infty} m(S) = 0.\#
\]

If \( m(S) > 0, \) then

\[
\infty = \sum_{k=1}^{\infty} m(S) = \sum_{k=1}^{\infty} m(q_k + S) = m\left( \bigcup_{k=1}^{\infty} q_k + S \right) \leq m([-1,2]) = 3.\#
\]

We thus see that it is impossible to find a measure \( m \) for arbitrary subsets of \( \mathbb{R} \) that satisfies the given four properties. The nonmeasurable set constructed here is called a \textit{Vitali set}. 
Remark: Since pathological sets like those encountered in the above example and in
the Banach–Tarski paradox rarely occur in practical applications of mathematics,
the standard approach to the problem of measure is to abandon the goal of assigning
a measure to every subset of $\mathbb{R}^d$, focusing instead on a certain subclass of “non-
pathological” subsets of $\mathbb{R}^d$, known as measurable sets.

1.B Elementary measure

Before we introduce Lebesgue measure and the associated Lebesgue integral, we will
first review the more elementary concept of Jordan measure. To formally define
Jordan measure, it is convenient to first introduce the concept of elementary measure,
which allows one to assign the notion of measure to elementary sets:

Definition: An interval $I$ is a subset of $\mathbb{R}$ of the form $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$,
$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$, $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$, $(a, b) = \{x \in \mathbb{R} : a < x < b\}$, where $a \leq b$ are real numbers.

Definition: The length of an interval $I$ is $|I| = b - a$. (We use the symbol $\doteq$ to
emphasize a definition, although the notation $:= $ is more common.)

Definition: A box in $\mathbb{R}^d$ is a Cartesian product $I_1 \times \ldots \times I_d$ of intervals $I_1 \ldots I_d$.

Definition: The measure of a box $B$ is $|B| = |I_1| \times \ldots \times |I_d|$.

Definition: An elementary set is a finite union of boxes.

Problem 1.2 (Boolean closure): If $E, F \subset \mathbb{R}^d$ are elementary sets, show that the
union $E \cup F$, the intersection $E \cap F$, the set theoretic difference $E \setminus F = \{x \in E : x \notin F\}$, and the symmetric difference $E \triangle F = (E \setminus F) \cup (F \setminus E)$ are also
elementary. If $x \in \mathbb{R}^d$, show that the translate $E + x = \{y + x : y \in E\}$ is also an
elementary set.

Lemma 1.1 (Measure of an elementary set): Let $E \subset \mathbb{R}^d$ be an elementary set.

(i) $E$ can be expressed as the finite union of disjoint boxes.

(ii) If $E$ is partitioned as a finite union $B_1 \cup \ldots \cup B_n$ of disjoint boxes, then the
quantity $m(E) \doteq |B_1| + \ldots + |B_n|$ is independent of the partition. In other
words, given any other partition $B'_1 \cup \ldots \cup B'_{n'}$ of $E$, one has $|B'_1| + \ldots + |B'_{n'}| =
|B_1| + \ldots + |B_n|$.

Proof:
(i) First consider the one-dimensional case $d = 1$. We can sort the $2n$ endpoints
of any finite collection of intervals $I_1, \ldots, I_n$ in ascending order, discarding
repetitions. Denote the open intervals between these endpoints, together with
the endpoints themselves (treated as degenerate intervals), by $J_1, \ldots, J_n'$. Each
interval $I_i$ can be expressed as a union of a finite subcollection of the disjoint
intervals $J_1, \ldots, J_n'$. The union $\bigcup_{i=1}^{n} I_i$ can thus be expressed as a finite union
of disjoint intervals. For $d > 1$, express $E = \bigcup_{i=1}^{n} B_i$ where $B_i = I_{i,1} \times \ldots \times I_{i,d}$.
For each $j = 1, \ldots, d$ we can decompose $I_{1,j}, \ldots, I_{n,j}$ as the union of disjoint
intervals. On taking the Cartesian product over $j = 1, \ldots, d$ we can then express
$E$ as a finite union of disjoint boxes.

(ii) Let $\# A$ denote the cardinality of a finite set $A$ and define $\frac{1}{N} \mathbb{Z} = \{ \frac{n}{N} : n \in \mathbb{Z} \}$.
The length $|I|$ of any interval $I$ can then be computed as
$$|I| = \lim_{N \to \infty} \frac{1}{N} \# \left( I \cap \frac{1}{N} \mathbb{Z} \right).$$
On taking the Cartesian product, we find that the measure of a box $B$ can be expressed as
$$|B| = \lim_{N \to \infty} \frac{1}{N^d} \# \left( B \cap \left( \frac{1}{N} \mathbb{Z} \right)^d \right).$$
The measure of an elementary set $E$ can thus be expressed as
$$m(E) = \lim_{N \to \infty} \frac{1}{N^d} \# \left( E \cap \left( \frac{1}{N} \mathbb{Z} \right)^d \right),$$
independent of its decomposition into disjoint boxes.

**Definition:** We refer to $m(E)$ in Lemma 1.1 as the *elementary measure* of $E$.

- The elementary measure of $[1, 2] \cup (3, 5)$ is 3.

**Remark:** The elementary measure $m(E)$ of an elementary set $E$ is non-negative and
satisfies the following properties:

1. $m(\emptyset) = 0$; \hspace{1cm} *nullity*

2. $m([a_1, b_1] \times \ldots \times [a_d, b_d]) = \prod_{k=1}^{d} (b_k - a_k)$ \hspace{1cm} *measure of a box*

3. $m(E + x) = m(E) \forall x \in \mathbb{R}^d$; \hspace{1cm} *translational invariance*

4. $m \left( \bigcup_{k=1}^{n} E_k \right) = \sum_{k=1}^{n} m(E_k)$ for disjoint elementary sets $E_k$. \hspace{1cm} *finite disjoint additivity*
Remark: As previously shown, these properties imply the *monotonicity* property

\[ E \subset F \Rightarrow m(E) \leq m(F) \]

for nested elementary sets \( E \) and \( F \).

**Problem 1.3:** Show that the above properties imply the *finite subadditivity* property

\[ m(E \cup F) \leq m(E) + m(F), \]

for arbitrary (not necessarily disjoint) elementary sets \( E \) and \( F \). By induction, deduce that

\[ m\left( \bigcup_{k=1}^{n} E_k \right) \leq \sum_{k=1}^{n} m(E_k) \]

for arbitrary elementary sets \( E_k \).

**Problem 1.4** (Uniqueness of elementary measure): Let \( d = 1 \). Let \( m' : \mathcal{E}(\mathbb{R}^d) \to [0, \infty) \) be a map from the collection \( \mathcal{E}(\mathbb{R}^d) \) of elementary subsets of \( \mathbb{R}^d \) to \( [0, \infty) \) that satisfies non-negativity, finite disjoint additivity, and translational invariance. Prove that there exists a constant \( c \in \mathbb{R} \) such that \( m'(E) = cm(E) \) for all elementary sets \( E \). Moreover, if we enforce the normalization \( m'([0,1]^d) = 1 \), show that \( m' = m \). Hint: Set \( c = m'([0,1]^d) \), and then compute \( m'([0,1/n]^d) \) for any positive integer \( n \).

**Problem 1.5:** Let \( d_1, d_2 \geq 1 \), and \( E_1 \subset \mathbb{R}^{d_1}, E_2 \subset \mathbb{R}^{d_2} \) be elementary sets. Show that \( E_1 \times E_2 \subset \mathbb{R}^{d_1+d_2} \) is elementary, with measure \( m^{d_1+d_2}(d_1 + d_2)(E_1 \times E_2) = m^{d_1}(E_1)m^{d_2}(E_2) \).
1.C Jordan measure

Definition: Let $S \subset \mathbb{R}^d$ be a bounded set. The Jordan inner measure $m_{*J}(S)$ of $S$ is

$$m_{*J}(S) = \sup_{E \subset S} m(E).$$

The Jordan outer measure $m^{*J}(S)$ of $S$ is

$$m^{*J}(S) = \inf_{E \supset S} m(E).$$

Definition: If $m_{*J}(S) = m^{*J}(S)$, we say that $S$ is Jordan measurable, and call $m(S) = m_{*J}(S) = m^{*J}(S)$ the Jordan measure of $S$. To emphasize the dimension $d$, we sometimes write $m(S)$ as $m^d(S)$.

Remark: The finite disjoint additivity and subadditivity properties of elementary measure allow us to rewrite the Jordan outer measure $m^{*J}(S)$ of $S$ as

$$m^{*J}(S) = \inf_{\bigcup_{k=1}^n B_k \supset S} \sum_{k=1}^n |B_k|.$$ 

Remark: Unbounded sets are not Jordan measurable (we say they have infinite Jordan inner and outer measure).

Remark: The following lemma shows that Jordan measurable sets are those sets that are “almost elementary” with respect to Jordan outer measure.

Lemma 1.2: Let $S \subset \mathbb{R}^d$ be bounded. The following are equivalent:

(i) $S$ is Jordan measurable

(ii) For every $\epsilon > 0$, there exist elementary sets $E$ and $F$ such that $E \subset S \subset F$ and $m(F \setminus E) < \epsilon$.

(iii) For every $\epsilon > 0$, there exists an elementary set $E$ such that $m^{*J}(E \triangle S) < \epsilon$.

Remark: Many common geometrical objects are Jordan measurable:

• A compact convex polytope in $\mathbb{R}^d$ formed by intersecting finitely many closed half spaces $\{x \in \mathbb{R}^d : a \cdot x \geq b\}$, where $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$;
• The region under the graph of a continuous non-negative function;

• The open Euclidean ball $B_r(c) = \{x \in \mathbb{R}^d : |x - c| < r\}$;

• The closed Euclidean ball $\overline{B}_r(c)$;

• The linear transformation of a Jordan measurable set.

Remark: Jordan measure inherits many of the properties of elementary measure.

Lemma 1.3: Let $S, T \subset \mathbb{R}^d$ be Jordan measurable sets. Then

(i) $S \cup T, S \cap T, S \setminus T, \text{ and } S \triangle T$ are Jordan measurable. Boolean closure

(ii) $m(S) \geq 0$; non-negativity

(iii) If $S$ and $T$ are disjoint then $m(S \cup T) = m(S) + m(T)$; finite disjoint additivity

(iv) If $S \subset T$, then $m(S) \leq m(T)$; monotonicity

(v) $m(S \cup T) \leq m(S) + m(T)$; finite subadditivity

(vi) For any $x \in \mathbb{R}^d$, $m(S + x) \exists = m(S)$. translational invariance

Problem 1.6:

(a) Show that a set $S$ and its topological closure $\overline{S}$ have the same Jordan outer measure.

(b) A set $S$ and its interior $S^\circ$ have the same Jordan outer measure.

(c) Show that a set $S$ is Jordan measurable iff its topological boundary $\partial S$ has Jordan outer measure zero. Use this result to prove that $\mathbb{Q} \cap [0, 1]$ is not measurable.
1.D Lebesgue measure

**Definition:** Let $S \subset \mathbb{R}^d$. The Lebesgue outer measure $m^*(S)$ of $S$ is

$$m^*(S) \doteq \inf_{\bigcup_{k=1}^{\infty} B_k \supset S} \sum_{k=1}^{\infty} |B_k|.$$  

**Remark:** The Lebesgue outer measure of a set is the infimal “cost” required to cover it by a countable union of boxes.

**Remark:** Since we can always convert a finite union of boxes to an infinite union by adding an infinite number of empty boxes, we see that $m^*(S) \leq m^J(S)$.

**Remark:** Observe that the Lebesgue outer measure can now be defined for unbounded sets.

- Let $S = \{x_0, x_1, \ldots\} \subset \mathbb{R}^d$ be a countable set. The Lebesgue outer measure of $S$ is easily seen to be zero: one simply covers $S$ by the degenerate boxes $\{x_0\}$, $\{x_1\}$, $\ldots$, each having measure zero. Alternatively given $\epsilon > 0$, one can cover each $x_k$ by a hypercube of sidelength $\epsilon/2^k$, leading to a total cost $\sum_{k=0}^{\infty} (\epsilon/2^k)^d = 2^d \epsilon^d$; on taking the infimum, the Lebesgue outer measure of $S$ is then seen to be zero. In contrast, the Jordan outer measure of a countable set can be large: $m^J(\mathbb{Q} \cap [0, R]) = R$ and $m^*J(\mathbb{Q}) = \infty$.

**Definition:** Given a sequence $\{x_k\}_{k=1}^{\infty}$ of non-negative extended real numbers, we define

$$\sum_{k=1}^{\infty} x_k = \sup_{F \text{ finite}} \sum_{n \in F} x_n,$$

which may be finite or infinite.

**Definition:** Given any collection $\{x_a\}_{a \in A}$ of non-negative extended real numbers indexed by an arbitrary set $A$, we define

$$\sum_{a \in A} x_a \doteq \sup_{F \text{ finite}} \sum_{a \in F} x_a.$$
Remark: One can relabel the collection in an arbitrary manner without affecting the sum. In particular:

**Theorem 1.1** (Tonelli’s theorem for series): Let \( \{x_{n,k}\}_{n,k \in \mathbb{N}} \) be a doubly infinite sequence of extended non-negative real numbers. Then

\[
\sum_{(n,k) \in \mathbb{N}^2} x_{n,k} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} x_{n,k} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} x_{n,k}.
\]

Remark: Note that this rearrangement does not hold when dealing with signed summands (cf. Riemann rearrangement theorem).

**Theorem 1.2** (Properties of Lebesgue outer measure):

(i) \( m^*(\emptyset) = 0 \); nullity

(ii) \( S \subset T \subset \mathbb{R}^d \Rightarrow m^*(S) \leq m^*(T) \); monotonicity

(iii) \( m^* \left( \bigcup_{k=1}^{\infty} S_k \right) \leq \sum_{k=1}^{\infty} m^*(S_k), \) where \( S_k \subset \mathbb{R}^d \). countable subadditivity

Proof:

(ii) Given \( \epsilon > 0 \), there exists a union of boxes \( B_k \) such that

\[
\bigcup_{k=1}^{\infty} B_k \supset T \supset S \text{ and } \sum_{k=1}^{\infty} |B_k| < m^*(T) + \epsilon.
\]

Thus

\[
m^*(S) \leq \sum_{k=1}^{\infty} |B_k| < m^*(T) + \epsilon.
\]

Since this holds for all \( \epsilon > 0 \), the only possibility is that \( m^*(S) \leq m^*(T) \).

(iii) Given \( \epsilon > 0 \), for each \( k \in \mathbb{N} \), there exists a countable union \( \bigcup_{n=1}^{\infty} B_{k,n} \) of boxes \( B_{k,n} \) containing \( S_k \) such that

\[
\sum_{n=1}^{\infty} |B_{k,n}| < m^*(S_k) + \frac{\epsilon}{2^k}.
\]

Since

\[
\bigcup_{k=1}^{\infty} S_k \subset \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} B_{k,n},
\]

...
we see that

\[ m^\ast \left( \bigcup_{k=1}^\infty S_k \right) \leq \sum_{k=1}^\infty \sum_{n=1}^\infty |B_{k,n}| < \sum_{k=1}^\infty \left( m^\ast(S_k) + \frac{\epsilon}{2^k} \right) = \sum_{k=1}^\infty m^\ast(S_k) + \sum_{k=1}^\infty \frac{\epsilon}{2^k} = \sum_{k=1}^\infty m^\ast(S_k) + \epsilon. \]

Thus \( m^\ast \left( \bigcup_{k=1}^\infty S_k \right) \leq \sum_{k=1}^\infty m^\ast(S_k). \)

**Definition**: Let \( X \) and \( Y \) be two nonempty subsets of \( \mathbb{R}^d \). The *distance* between \( X \) and \( Y \) is

\[ \text{dist}(X,Y) = \inf\{|x-y|: x \in X, y \in Y\}. \]

**Definition**: The *diameter* of a nonempty set \( S \) is \( \sup\{|x-y|: x, y \in S\} \).

**Definition**: Two sets \( S, T \subset \mathbb{R}^d \) are separated if \( \text{dist}(S,T) > 0 \).

**Problem 1.7**: Prove that compact disjoint sets in \( \mathbb{R}^d \) are separated.

**Lemma 1.4** (Finite additivity for separated sets): *Let \( S \) and \( T \) be separated sets. Then \( m^\ast(S \cup T) = m^\ast(S) + m^\ast(T) \).*

Proof: From subadditivity, we know that \( m^\ast(S \cup T) \leq m^\ast(S) + m^\ast(T) \), so it suffices to prove \( m^\ast(S) + m^\ast(T) \leq m^\ast(S \cup T) \). This is trivial if \( S \cup T \) has infinite Lebesgue outer measure. Otherwise \( S \cup T \) has finite Lebesgue outer measure and, by monotonicity, so do \( S \) and \( T \).

Given \( \epsilon > 0 \), without loss of generality we can cover \( S \cup T \) by a countable collection of boxes \( B_1, B_2, \ldots \) with diameter less than \( \text{dist}(S,T) > 0 \) such that \( \sum_{k=1}^\infty |B_k| < m^\ast(S \cup T) + \epsilon \). By construction, each box intersects at most one of \( S \) and \( T \). We can therefore split this collection into two countable subcollections \( B'_1, B'_2, \ldots \) and \( B''_1, B''_2, \ldots \) that cover \( S \) and \( T \), respectively. Then

\[ m^\ast(S) \leq \sum_{k=1}^\infty |B'_k| \]

and

\[ m^\ast(T) \leq \sum_{k=1}^\infty |B''_k|. \]

On summing, we find that

\[ m^\ast(S) + m^\ast(T) \leq \sum_{k=1}^\infty |B'_k| + \sum_{k=1}^\infty |B''_k| = \sum_{k=1}^\infty |B_k|. \]
Thus
\[ m^*(S) + m^*(T) < m^*(S \cup T) + \epsilon. \]
We have thus obtained the desired result
\[ m^*(S) + m^*(T) \leq m^*(S \cup T). \]

**Lemma 1.5** (Outer measure of elementary set): Let \( E \) be an elementary set. Then the Lebesgue outer measure \( m^*(E) \) and the elementary measure \( m(E) \) are equal.

Proof: Since \( m^*(E) \leq m^*(J) = m(E) \), we only need to establish that \( m(E) \leq m^*(E) \). We first show this in the case where \( E \) is closed. Given \( \epsilon > 0 \), there exists a countable family \( B_1, B_2, \ldots \) of boxes that cover \( E \) such that
\[ \sum_{k=1}^{\infty} |B_k| < m^*(E) + \epsilon. \]
Each box \( B_k \) can be enclosed within an open box \( B_k' \) such that \( |B_k'| < |B_k| + \epsilon/2^k \). We have thus constructed an open cover \( \bigcup_{k=1}^{\infty} B_k' \) of the closed set \( E \), with
\[ \sum_{k=1}^{\infty} |B_k'| < \sum_{k=1}^{\infty} \left( |B_k| + \frac{\epsilon}{2^k} \right) = \sum_{k=1}^{\infty} |B_k| + \epsilon < m^*(E) + 2\epsilon. \]
By the Heine-Borel theorem, \( E \) is compact and therefore has a finite subcover \( \bigcup_{k=1}^{n} B_k' \) for some \( n \in \mathbb{N} \). The finite subadditivity of elementary measure then yields
\[ m(E) \leq \sum_{k=1}^{n} |B_k'| \leq \sum_{k=1}^{\infty} |B_k'| < m^*(E) + 2\epsilon. \]

Then \( m(E) \leq m^*(E) \) and hence \( m(E) = m^*(E) \) for any closed elementary set \( E \).

If \( E \) is not closed, express \( E \) as a finite union \( \bigcup_{k=1}^{n} B_k \) of disjoint boxes. Given \( \epsilon > 0 \), for every \( k \in 1, \ldots, n \) there is a closed sub-box \( B_k' \) of \( B_k \) such that \( |B_k| - \epsilon/n \leq |B_k'| \). Then
\[ m(E) - \epsilon = \sum_{k=1}^{n} (|B_k| - \epsilon/n) \leq \sum_{k=1}^{n} |B_k'| = m \left( \bigcup_{k=1}^{n} B_k' \right) = m^* \left( \bigcup_{k=1}^{n} B_k' \right) \leq m^*(E), \]
using our result for the closed elementary set \( \bigcup_{k=1}^{n} B_k' \subset E \) and the monotonicity of the Lebesgue outer measure. Thus \( m(E) \leq m^*(E) \).

**Remark:** We have seen that the Lebesgue outer measure of a countable set is zero. Since the elementary measure of \([0, 1]\) is 1, the above lemma then establishes that the Lebesgue outer measure of \([0, 1]\) is also 1. This proves that the real numbers are uncountable.
1.D. LEBESGUE MEASURE

Remark: Since the Lebesgue outer measure $m^*(E)$ of an elementary set $E$ is $m(E)$, we observe from monotonicity that for any $S \subset \mathbb{R}^d$,

$$m^*_J(S) \doteq \sup_{E \subset S} m(E) = \sup_{E \text{ elementary}} m^*(E) \leq \sup_{E \subset S} m^*(E) = m^*(S).$$

Thus

$$m^*_J(S) \leq m^*(S) \leq m^*_J(S).$$

Remark: Not every bounded open or compact set is Jordan measurable. For example, enumerate the countable set $\mathbb{Q} \cap [0, 1]$ as $q_1, q_2, \ldots$. Given $\epsilon > 0$, the Lebesgue outer measure of the open union

$$U \doteq \bigcup_{k=1}^{\infty} \left(q_k - \frac{\epsilon}{2^k}, q_k + \frac{\epsilon}{2^k}\right)$$

is

$$m^*(U) \leq \sum_{k=1}^{\infty} \frac{2\epsilon}{2^k} = 2\epsilon.$$

In contrast, since $[0, 1] \subset U$ ($U$ is dense in $[0, 1]$), we see from Problem 1.6(a) that

$$1 = m^*_J([0, 1]) \leq m^*_J(U) = m^*_J(U) \leq 1.$$  

For $\epsilon = 1/3$ (say), we see that the Lebesgue and Jordan outer measures of the open set $U$ disagree. Moreover, from the previous remark, we see that $m^*_J(U) \leq m^*(U) = 0$, which implies that $U$ is not Jordan measurable. Likewise, the complement of $U$ in $[-1, 2]$ is compact but not Jordan measurable.

Definition: Two boxes are almost disjoint if their interiors are disjoint.

- $[0, 1]$ and $[1, 2]$ are almost disjoint.

Remark: Since a box has the same elementary measure as its interior, the finite additivity property

$$m \left( \bigcup_{k=1}^{n} B_k \right) = \sum_{k=1}^{n} |B_k|$$

also holds for almost disjoint boxes $B_k$.

Lemma 1.6 (Countable unions of almost disjoint boxes): Let $S$ be a countable union of almost disjoint boxes $B_1, B_2, \ldots$. Then

$$m^*(S) = \sum_{k=1}^{\infty} |B_k|.$$
Proof: Using countable subadditivity and Lemma 1.5, we find
\[ m^*(S) \leq \sum_{k=1}^{\infty} m^*(B_k) = \sum_{k=1}^{\infty} |B_k|. \]

Moreover, for each \( n \in \mathbb{N} \), the elementary set \( \bigcup_{k=1}^{n} B_k \) is contained in \( S \). Using Lemma 1.5 again and monotonicity, we find that
\[ \sum_{k=1}^{n} |B_k| = m\left( \bigcup_{k=1}^{n} B_k \right) = m^*\left( \bigcup_{k=1}^{n} B_k \right) \leq m^*(S). \]

On letting \( n \to \infty \), we obtain
\[ \sum_{k=1}^{\infty} |B_k| \leq m^*(S), \]
which establishes the desired result.

**Remark:** We thus see that \( \mathbb{R}^d \) has an infinite Lebesgue outer measure.

**Problem 1.8:** If a set can be expressed as the countable union of almost disjoint boxes, show that its Lebesgue outer measure is equal to its Jordan inner measure.

**Definition:** Let \( n \in \mathbb{Z} \). A closed dyadic cube with sidelength \( 2^{-n} \) has the form
\[ \left[ \frac{i_1}{2^n}, \frac{i_1 + 1}{2^n} \right] \times \ldots \times \left[ \frac{i_d}{2^n}, \frac{i_d + 1}{2^n} \right] \]
for integers \( i_1, i_2, \ldots, i_d \).

**Remark:** The set of closed dyadic cubes of a fixed sidelength \( 2^{-n} \) are almost disjoint and cover all of \( \mathbb{R}^d \).

**Remark:** Each closed dyadic cube in \( \mathbb{R}^d \) of sidelength \( 2^{-n} \) has \( 2^d \) children of sidelength \( 2^{-n-1} \).

**Definition:** Given a collection \( Q \) of closed dyadic cubes, we say that a cube is maximal if it is not contained in any other cubes in \( Q \).

**Lemma 1.7:** Every open set can be expressed as a countable union of almost disjoint closed cubes.

Proof: Recall that around every point \( x \) of an open set \( S \) there is an open ball entirely contained within \( S \). Since each open ball contains a closed dyadic cube that includes \( x \), an open set is the (countable) union of a collection \( Q \) of dyadic cubes contained within it. Let us restrict \( n \geq 0 \) to enforce a maximum cube sidelength of 1 and let \( Q^* \) denote the collection of cubes that are maximal in \( Q \). Note that every cube in \( Q \) is contained in exactly one maximal cube and that any two such maximal cubes are almost disjoint. We thus see that \( S \) can be expressed as a countable union of (almost disjoint) maximal cubes.
Remark: From Prob 1.8, we thus see that for open sets, the Lebesgue outer measure is equal to the Jordan inner measure and corresponds to the total measure of any countable partitioning of the set into almost disjoint boxes.

Q. Can we express the Lebesgue outer measure of an arbitrary set in terms of open sets?

Lemma 1.8 (Outer regularity): Let $S \subset \mathbb{R}^d$ be an arbitrary set. Then

$$m^*(S) = \inf_{U \supset S \text{ open}} m^*(U).$$

Proof: It follows immediately from monotonicity that

$$m^*(S) \leq \inf_{U \supset S \text{ open}} m^*(U).$$

The other direction

$$\inf_{U \supset S \text{ open}} m^*(U) \leq m^*(S)$$

holds trivially if $m^*(S)$ is infinite. If $m^*(S)$ is finite, given $\epsilon > 0$, there exists a countable collection $B_1, B_2, \ldots$ of boxes that cover $S$, with

$$\sum_{k=1}^{\infty} |B_k| < m^*(S) + \epsilon.$$

Each box $B_k$ can be enclosed within an open box $B'_k$ such that $|B'_k| < |B_k| + \epsilon/2^k$, with

$$\sum_{k=1}^{\infty} |B'_k| < \sum_{k=1}^{\infty} \left( |B_k| + \frac{\epsilon}{2^k} \right) = \sum_{k=1}^{\infty} |B_k| + \epsilon < m^*(S) + 2\epsilon.$$

From countable subadditivity, we see that

$$m^*\left( \bigcup_{k=1}^{\infty} B'_k \right) \leq \sum_{k=1}^{\infty} |B'_k| < m^*(S) + 2\epsilon,$$

from which we deduce that

$$\inf_{U \supset S \text{ open}} m^*(U) < m^*(S) + 2\epsilon.$$

The desired result then follows. \qed

Remark: Outer regularity motivates the following definition.
Definition: A set $S \subset \mathbb{R}^d$ is said to be Lebesgue measurable if for every $\epsilon > 0$, there exists an open set $U \subset \mathbb{R}^d$ containing $S$ such that $m^*(U \setminus S) < \epsilon$.

Definition: If $S$ is Lebesgue measurable, we refer to $m(S) = m^*(S)$ as the Lebesgue measure of $S$. To emphasize the dimension $d$, we sometimes write $m(S)$ as $m^d(S)$.

Remark: A Lebesgue measurable set is one that can be contained efficiently within an open set (with respect to Lebesgue measure).

Lemma 1.9 (Lebesgue measurability of compact sets): Every compact set is Lebesgue measurable.

Proof: Let $S$ be a compact set. Given $\epsilon > 0$, Lemma 1.8 there exists an open set $U$ containing $S$ such that $m^*(U) < m^*(S) + \epsilon$. The set $U \setminus S = U \cap S^c$ is open and so by Lemma 1.7 is the countable union $\bigcup_{k=1}^{\infty} Q_k$ of almost disjoint closed cubes $Q_k$. By Lemma 1.6, $m^*(U \setminus S) = \sum_{k=1}^{\infty} |Q_k|$. The finite union $\bigcup_{k=1}^{n} Q_k$ for every $n \in \mathbb{N}$ of closed cubes is itself closed and is disjoint from the compact set $S$. Since compact disjoint sets in $\mathbb{R}^d$ are separated, it follows from Lemma 1.4 and monotonicity that

$$m^*(S) + m^* \left( \bigcup_{k=1}^{n} Q_k \right) = m^* \left( S \cup \bigcup_{k=1}^{n} Q_k \right) \leq m^*(U) < m^*(S) + \epsilon.$$ 

Since for each $n \in \mathbb{N}$, we have $\sum_{k=1}^{n} |Q_k| = m^*(\bigcup_{k=1}^{n} Q_k) < \epsilon$, it follows that $m^*(U \setminus S) = \sum_{k=1}^{\infty} |Q_k| \leq \epsilon$. Hence $S$ is Lebesgue measurable.

Definition: A null set is a set of Lebesgue outer measure zero.

Lemma 1.10 (Measurable Lebesgue sets): The following sets are Lebesgue measurable:

(i) An open set.

(ii) A closed set.

(iii) A null set.

(iv) The empty set.

(v) The complement of a Lebesgue measurable subset of $\mathbb{R}^d$.

(vi) A countable union of Lebesgue measurable sets.

(vii) A countable intersection of Lebesgue measurable sets.

Proof: Claims (i), (iii), and (iv) follow directly from the definition of Lebesgue measurability.
(vi) Given $\epsilon > 0$, let $S_1, S_2, \ldots$ be a sequence of Lebesgue measurable sets. For each $k \in \mathbb{N}$, $S_k$ is contained in an open set $U_k$ such that $m^*(U_k \setminus S_k) < \epsilon/2^k$. The Lebesgue measurability of $\bigcup_{k=1}^{\infty} S_k$ follows from the openness of $\bigcup_{k=1}^{\infty} U_k$ and countable subadditivity: $m^*(\bigcup_{k=1}^{\infty} U_k \setminus \bigcup_{k=1}^{\infty} S_k) \leq \sum_{k=1}^{\infty} m^*(U_k \setminus S_k) < \sum_{k=1}^{\infty} \epsilon/2^k = \epsilon$.

(ii) Given a closed set $S \subset \mathbb{R}^d$, we can express it as a countable union of compact sets: $S = \bigcup_{k=1}^{\infty} B_k(0) \cap S$. The result then follows from Lemma 1.9 and (vi).

(v) If $S$ is Lebesgue measurable then for each $n \in \mathbb{N}$ there exists an open set $U_n$ containing $S$ such that $m^*(U_n \setminus S) < 1/n$. Let $F = \bigcup_{n=1}^{\infty} U_n^c$. For each $n \in \mathbb{N}$, we note that $S^c \cap F \subset S^c \setminus U_n^c$ and hence, using monotonicity, $m^*(S^c \cap F) \leq m^*(S^c \setminus U_n^c) < 1/n$, noting that $S^c \setminus U_n^c = U_n \setminus S$. Hence $m^*(S^c \cap F) = 0$, so that $S^c = F \cup (S^c \setminus F)$ is the union of a countably many closed sets $U_n^c$ and a set of Lebesgue outer measure zero. By (ii), (iii), and (vi), $S^c$ is Lebesgue measurable.

(vii) Apply DeMorgan’s laws to (vi), using (v) twice.

Definition: A Boolean algebra is an algebraic structure that characterizes both set and logic operations: a subset $Y$ of a set $X$ is associated with a collection of bits indexed by $x \in X$ as 1 or 0 according to whether or not $x \in Y$. A Boolean algebra is thus analogous to the field of sets, which is any nonempty collection of subsets of a given set closed under finite union, finite intersection, and complement operations.

Remark: Properties (iv), (v), and (vi) of Lemma 1.10 state that the set of Lebesgue measurable sets is closed under countably many Boolean operations and thus forms a $\sigma$-algebra, a generalization of a Boolean algebra that requires closure under countable unions.\(^1\)

\(^1\)The $\sigma$ (the Greek letter corresponding to the initial $s$ in the German word “Summe” originally used to denote union) in $\sigma$-algebra denotes closure under countable union. Likewise, $\delta$-algebra is closed under countable intersections ($\delta$ is the Greek letter corresponding to the initial $d$ in the German word “Durchschnitt” for intersection).
Lemma 1.11 (Characterization of measurability): Let $S \subset \mathbb{R}^d$. The following are equivalent:

(i) $S$ is Lebesgue measurable.

(ii) Given $\epsilon > 0$, there exists an open set $U_\epsilon$ containing $S$ with $m^*(U_\epsilon \setminus S) < \epsilon$. (outer open approximation);

(iii) Given $\epsilon > 0$, there exists an open set $U_\epsilon$ with $m^*(U_\epsilon \triangle S) < \epsilon$. (almost open);

(iv) Given $\epsilon > 0$, there exists a closed set $F_\epsilon$ contained in $S$ with $m^*(S \setminus F_\epsilon) < \epsilon$. (inner closed approximation);

(v) Given $\epsilon > 0$, there exists a closed set $F_\epsilon$ with $m^*(F_\epsilon \triangle S) < \epsilon$. (almost closed);

(vi) Given $\epsilon > 0$, there exists a Lebesgue measurable set $S_\epsilon$ with $m^*(S_\epsilon \triangle S) < \epsilon$. (almost measurable).

Proof:

(i) $\Rightarrow$ (ii) This is the definition of Lebesgue measurability.

(ii) $\Rightarrow$ (iii) $U_\epsilon \supset S \Rightarrow U_\epsilon \triangle S = U_\epsilon \setminus S$.

(iii) $\Rightarrow$ (vi) This follows from Lemma 1.10.

(i) $\Rightarrow$ (iv) By Lemma 1.10 (v), $S^c$ is Lebesgue measurable: given $\epsilon > 0$, there exists $U_\epsilon$ such that $S^c \subset U_\epsilon$, with $m^*(U_\epsilon \setminus S^c) < \epsilon$. Hence $S \supset U_\epsilon^c$, where $U_\epsilon^c$ is closed. Since $U_\epsilon \setminus S^c = U_\epsilon \cap S = S \cap U_\epsilon = S \setminus U_\epsilon^c$, we see that $m^*(S \setminus U_\epsilon^c) < \epsilon$.

(iv) $\Rightarrow$ (v) $S \supset F_\epsilon \Rightarrow F_\epsilon \triangle S = S \setminus F_\epsilon$.

(v) $\Rightarrow$ (vi) This follows from Lemma 1.10.

(vi) $\Rightarrow$ (i) First, notice for any sets $A$, $B$, and $S$ that $(A \cup B) \triangle S \subset (A \triangle S) \cup B$.

Given $\epsilon > 0$, there exists a measurable set $S_\epsilon$ with $m^*(S_\epsilon \triangle S) < \epsilon$. Since $S_\epsilon$ is measurable, it can be contained in an open set $U_\epsilon$ such that $m^*(U_\epsilon \setminus S_\epsilon) < \epsilon$. Then

$$U_\epsilon \triangle S \subset (S_\epsilon \triangle S) \cup (U_\epsilon \setminus S_\epsilon),$$

monotonicity and subadditivity then lead to

$$m^*(U_\epsilon \triangle S) \leq m^*(S_\epsilon \triangle S) + m^*(U_\epsilon \setminus S_\epsilon) < \epsilon + \epsilon = 2\epsilon.$$

By outer regularity, there exists an open set $V_\epsilon$ containing $S \setminus U_\epsilon$ with

$$m^*(V_\epsilon) \leq m^*(S \setminus U_\epsilon) + \epsilon \leq m^*(S \triangle U_\epsilon) + \epsilon < 2\epsilon + \epsilon = 3\epsilon.$$
Finally, we note that the open set $U_\epsilon \cup V_\epsilon$ contains $S$ and that by monotonicity and subadditivity,

$$m^*((U_\epsilon \cup V_\epsilon) \setminus S) = m^*((U_\epsilon \cup V_\epsilon) \triangle S) \leq m^*(U_\epsilon \triangle S) + m^*(V_\epsilon) < 2\epsilon + 3\epsilon = 5\epsilon,$$

as desired.

**Remark:** We thus see that every measurable set $S$ can be expressed as the union of a closed set $F \subset S$ and a set $S \setminus F$ of arbitrarily small measure. Likewise, there exists a set of arbitrarily small measure whose union with $S$ is open.

**Lemma 1.12** (Properties of Lebesgue measure):

(i) $m(\emptyset) = 0$; \hspace{1cm} \textit{(nullity)}

(ii) If $S_1, S_2, \ldots \subset \mathbb{R}^d$ is a countable sequence of disjoint Lebesgue measurable sets, then $m(\bigcup_{k=1}^{\infty} S_k) = \sum_{k=1}^{\infty} m(S_k)$. \hspace{1cm} \textit{countable additivity}

**Proof:**

(i) We note that $m^*(\emptyset \setminus \emptyset) = m^*(\emptyset) = 0$.

(ii) First suppose that all of the sets $S_k$ are compact and let $n \in \mathbb{N}$. Since compact disjoint sets in $\mathbb{R}^d$ are separated, we see using Lemma 1.4 and monotonicity that

$$\sum_{k=1}^{n} m(S_k) = m\left(\bigcup_{k=1}^{n} S_k\right) \leq m\left(\bigcup_{k=1}^{\infty} S_k\right),$$

on replacing $m^*$ with $m$, in view of Lemma 1.10. As $n \to \infty$, we see that

$$\sum_{k=1}^{\infty} m(S_k) \leq m\left(\bigcup_{k=1}^{\infty} S_k\right).$$

Also, from countable subadditivity, we have

$$m\left(\bigcup_{k=1}^{\infty} S_k\right) \leq \sum_{k=1}^{\infty} m(S_k).$$

The result then follows.

If $S_k$ are bounded but not necessarily compact, we know from Lemma 1.11 that they can each be expressed as the union of a compact set $K_k$ and a set of arbitrarily small measure:

$$m(S_k) < m(K_k) + \frac{\epsilon}{2^k},$$
so that
\[ \sum_{k=1}^{\infty} m(S_k) \leq \sum_{k=1}^{\infty} m(K_k) + \epsilon. \]

But from the compact case and monotonicity, we know that
\[ \sum_{k=1}^{\infty} m(K_k) = m\left( \bigcup_{k=1}^{\infty} K_k \right) \leq m\left( \bigcup_{k=1}^{\infty} S_k \right), \]
from which we deduce
\[ \sum_{k=1}^{\infty} m(S_k) \leq m\left( \bigcup_{k=1}^{\infty} S_k \right) + \epsilon \]
for every \( \epsilon > 0 \). The result then follows from countable subadditivity:
\[ m\left( \bigcup_{k=1}^{\infty} S_k \right) \leq \sum_{k=1}^{\infty} m(S_k). \]

Finally, if the sets \( S_k \) are not bounded or closed, we decompose each of them as a countable union of disjoint Lebesgue measurable sets, using the bounded annuli \( A_m \equiv B_m(0) \setminus B_{m-1}(0) \) for \( m \in \mathbb{N} \):
\[ S_k = \bigcup_{m=1}^{\infty} S_k \cap A_m. \]
The bounded case then yields
\[ m(S_k) = \sum_{m=1}^{\infty} m(S_k \cap A_m). \]
We can similarly decompose \( \bigcup_{k=1}^{\infty} S_k \) as a countable union of the disjoint bounded measurable sets \( S_k \cap A_m \) over \( (k, m) \in \mathbb{N} \times \mathbb{N} \) to obtain
\[ m\left( \bigcup_{k=1}^{\infty} S_k \right) = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} m(S_k \cap A_m) = \sum_{k=1}^{\infty} m(S_k). \]

**Problem 1.9** (Monotone convergence theorem for Lebesgue measurable sets):

(i) *(Upward monotone convergence)* Let \( S_1 \subset S_2 \subset \ldots \) be a countable increasing sequence of Lebesgue measurable subsets of \( \mathbb{R}^d \). Show that \( m(\bigcup_{k=1}^{\infty} S_k) = \lim_{n \to \infty} m(S_n) \). Hint: Express \( \bigcup_{k=1}^{\infty} S_k \) in terms of the lacuna \( L_n = S_n \setminus \bigcup_{k=1}^{n-1} S_k \).

(ii) *(Downward monotone convergence)* Let \( S_1 \supset S_2 \supset \ldots \) be a countable decreasing sequence of Lebesgue measurable subsets of \( \mathbb{R}^d \). If at least one of the \( m(S_k) \) is finite, show that \( m(\bigcap_{k=1}^{\infty} S_k) = \lim_{n \to \infty} m(S_n) \).

(iii) Show that one cannot drop the assumption that at least one of the \( m(S_m) \) is finite.
Definition: A sequence \( \{S_n\}_{n=1}^{\infty} \) of sets in \( \mathbb{R}^d \) converges pointwise to another set \( S \) in \( \mathbb{R}^d \) if the indicator functions \( 1_{S_n} \) converge pointwise to \( 1_S \).

Definition: Recall that

\[
\limsup_{n \to \infty} x_n \doteq \limsup_{n \to \infty} \sup_{k \geq n} x_k \\
\liminf_{n \to \infty} x_n \doteq \liminf_{n \to \infty} \inf_{k \geq n} x_k.
\]

In the extended \( d \)-dimensional reals \( \mathbb{R}^d \), a sequence \( \{x_n\} \) converges iff \( \limsup_{n \to \infty} x_n = \liminf_{n \to \infty} x_n \).

Problem 1.10: Suppose \( S_n \subset \mathbb{R}^d, n = 1, 2, \ldots \) are Lebesgue measurable sets that converge pointwise to a set \( S \).

(i) Show that \( S \) is Lebesgue measurable. Hint: use the fact that \( 1_S(x) = \liminf_{n \to \infty} 1_{S_n}(x) = \limsup_{n \to \infty} 1_{S_n}(x) \) to write \( S \) in terms of countable unions and intersections of \( S_n \).

(ii) (Dominated convergence theorem) Suppose that the \( S_n \) are all contained in another Lebesgue measurable set \( F \) of finite measure. Show that \( m(S_n) \) converges to \( m(S) \). Hint: use the upward and downward monotone convergence theorems.

(iii) Give a counterexample to show that the dominated convergence theorem fails if the \( S_n \) are not contained in a set of finite measure, even if we assume that the \( m(S_n) \) are all uniformly bounded.

Problem 1.11: Let \( S \subset \mathbb{R}^d \). Show that \( S \) is contained in a Lebesgue measurable set of measure \( m^*(S) \).

Problem 1.12: Let \( S \subset \mathbb{R}^d \). Provide a counterexample that establishes that the claim

\[
m^*(S) = \sup_{U \subset S, U \text{ open}} m^*(U)
\]

if false.

Problem 1.13: (Inner regularity). Let \( S \subset \mathbb{R}^d \) be Lebesgue measurable. Show that

\[
m(S) = \sup_{K \subset S, K \text{ compact}} m(K).
\]
Problem 1.14: (Characterization of finite measurability) Let $S \subset \mathbb{R}^d$. Given $\epsilon > 0$, show that the following are equivalent:

(i) $S$ is Lebesgue measurable with finite measure;

(ii) There exists an open set $U_\epsilon$ of finite measure containing $S$ with $m^*(U_\epsilon \setminus S) < \epsilon$; (outer open approximation)

(iii) There exists a bounded open set $U_\epsilon$ with $m^*(U_\epsilon \triangle S) < \epsilon$; (almost open bounded)

(iv) There exists a compact set $K_\epsilon$ contained in $S$ with $m^*(S \setminus K_\epsilon) < \epsilon$; (inner compact approximation)

(v) There exists a compact set $K_\epsilon$ with $m^*(K_\epsilon \triangle S) < \epsilon$; (almost compact)

(vi) There exists a bounded Lebesgue measurable set $S_\epsilon$ with $m^*(S_\epsilon \triangle S) < \epsilon$; (almost bounded measurable)

(vii) There exists a Lebesgue measurable set $S_\epsilon$ with finite measure such that $m^*(S_\epsilon \triangle S) < \epsilon$; (almost finite measure)

(viii) There exists an elementary set $E_\epsilon$ such that $m^*(E_\epsilon \triangle S) < \epsilon$; (almost elementary)

(ix) There exists a finite union $F_\epsilon$ of closed dyadic cubes such that $m^*(F_\epsilon \triangle S) < \epsilon$. (almost dyadic)

Problem 1.15: (Carathéodory criterion, one direction) Let $S \subset \mathbb{R}^d$. Show that the following are equivalent.

(i) $S$ is Lebesgue measurable;

(ii) For every elementary set $E$,

$$m(E) = m^*(E \cap S) + m^*(E \cap S^c);$$

(iii) For every box $B$,

$$m(B) = m^*(B \cap S) + m^*(B \cap S^c).$$

Definition: Let $S \subset \mathbb{R}^d$ be a bounded set contained within an elementary set $E$. The Lebesgue inner measure of $S$ is

$$m_*(S) = m(E) - m^*(E \setminus S).$$
Problem 1.16:

(i) Show that the definition of Lebesgue inner measure is well defined in that it does not depend on the choice of elementary set $E$.

(ii) Show that $m_*(S) \leq m^*(S)$ and that equality hold iff $S$ is Lebesgue measurable.

**Definition:** A $G_\delta$ set is a countable intersection of open sets.

**Definition:** An $F_\sigma$ set is a countable union of closed sets.

**Remark:** Note that a $G_\delta$ set need not be open and a $F_\sigma$ set need not be closed.

Problem 1.17: Show that the following are equivalent:

(i) $S$ is Lebesgue measurable;

(ii) $S$ is the difference of a $G_\delta$ set and a null set;

(iii) $S$ is the union of an $F_\sigma$ set and a null set.

Problem 1.18: (Translation invariance)

If $S \subset \mathbb{R}^d$ is Lebesgue measurable, show that $S + x$ is Lebesgue measurable for any $x \in \mathbb{R}^d$, with $m(S + x) = m(S)$.

1.E The Lebesgue integral

**Definition:** Let $\{c_k\}_{k=1}^{\infty}$ be a sequence of elements of the set of complex numbers $\mathbb{C}$. We say that $\{c_k\}_{k=1}^{\infty}$ converges absolutely or is absolutely summable if

$$\sum_{k=1}^{\infty} |c_k| < \infty.$$

**Remark:** Recall that the partial sums $\sum_{k=1}^{n} c_k$ of an absolutely convergent sequence converge to a finite number. Furthermore, one can rearrange the terms of an absolutely convergent sequence without affecting its sum, defined in terms of its real and imaginary parts as

$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \text{Re} c_k + i \sum_{k=1}^{\infty} \text{Im} c_k,$$

for complex $c_k$, where for real $c_k$,

$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} c_k^+ - \sum_{k=1}^{\infty} c_k^-,$$

with $c_k^+ = \max(c_k, 0)$ and $c_k^- = \max(-c_k, 0)$. 

**Definition:** A (complex-valued) simple function $f : \mathbb{R}^d \to \mathbb{C}$ is a finite linear combination

$$f = \sum_{k=1}^{n} c_k 1_{S_k}.$$ 

of indicator functions $1_{S_k}$ of Lebesgue measurable sets $S_k \subset \mathbb{R}^d$ for $c_k \in \mathbb{C}$, where $k = 1, \ldots, n$ and $n \in \mathbb{N}$. If $f : \mathbb{R}^d \to [0, \infty]$ and $c_K \in [0, \infty]$, we say that $f$ is an unsigned simple function.

**Definition:** Denote the complex vector space of simple functions over $\mathbb{R}^d$ as $\text{Simp}(\mathbb{R}^d)$.

**Definition:** Denote the space of unsigned simple functions over $\mathbb{R}^d$ as $\text{Simp}^+(\mathbb{R}^d)$.

**Remark:** In addition to the usual closure properties of a vector space, $\text{Simp}(\mathbb{R}^d)$ is also closed under the pointwise product $f, g \mapsto fg$ and complex conjugation $f \mapsto \overline{f}$, making $\text{Simp}(\mathbb{R}^d)$ a commutative $*$-algebra (or involutive algebra). The space $\text{Simp}^+(\mathbb{R}^d)$ is a $[0, \infty]$-module: it is closed under addition and under multiplication by elements of $[0, \infty]$.

**Remark:** Although we don’t require the Lebesgue measurable sets $S_1, \ldots, S_n$ to be disjoint, we can achieve this by noting that they partition $\mathbb{R}^d$ into $2^n$ measurable sets, each of which is an intersection of $S_1, \ldots S_n$ and their complements in $\mathbb{R}^d$.

**Definition:** Denote

$$\int_{\mathbb{R}^d} 1_S = m(S).$$

**Definition:** If $f = \sum_{k=1}^{n} c_k 1_{S_k}$ is an unsigned simple function, define the simple integral

$$\text{Simp} \int_{\mathbb{R}^d} f = \sum_{k=1}^{n} c_k m(S_k).$$

**Lemma 1.13** (Well-definedness of the simple integral): Let $n, n' \in \mathbb{N}$, $c_1, \ldots, c_n, c'_1, \ldots, c'_{n'} \in [0, \infty]$ and $S_1, \ldots S_n, S'_1, \ldots S'_{n'} \subset \mathbb{R}^d$ be Lebesgue measurable sets such that

$$\sum_{k=1}^{n} c_k 1_{S_k} = \sum_{k=1}^{n'} c'_k 1_{S'_k}.$$ 

Then

$$\sum_{k=1}^{n} c_k m(S_k) = \sum_{k=1}^{n'} c'_k m(S'_k).$$

Proof: The $k + k'$ sets $S_k$ and $S'_k$ partition $\mathbb{R}^d$ into $2^{k+k'}$ disjoint sets. The result then follows from the finite additivity of Lebesgue measure (for details see Tao, page 52, noting that $1_{A_i}$ should be replaced by $1_{A_{j_i}}$, etc.).
Definition: A property \( P(x) \) of a point \( x \in \mathbb{R}^d \) is said to hold almost everywhere (or a.e.) in \( \mathbb{R}^d \) if the set of \( x \in R^d \) on which \( P(x) \) fails has Lebesgue measure zero (i.e. \( P \) is true outside of a null set).

Definition: Two functions \( f \) and \( g \) on \( \mathbb{R}^d \) are said to agree almost everywhere if \( f(x) = g(x) \) almost everywhere in \( \mathbb{R}^d \).

Definition: The support of a function \( f : \mathbb{R}^d \to \mathbb{C} \) or \( f : \mathbb{R}^d \to [0, \infty] \) is the set \( \{ x \in \mathbb{R}^d : f(x) \neq 0 \} \).

Problem 1.19: (Properties of the simple unsigned integral). Let \( f, g : \mathbb{R}^d \to [0, \infty] \) be simple unsigned functions. Then

(i) \( \text{Simp} \int_{\mathbb{R}^d} (f + g) = \text{Simp} \int_{\mathbb{R}^d} f + \text{Simp} \int_{\mathbb{R}^d} g \)

and

\( \text{Simp} \int_{\mathbb{R}^d} cf = c \text{Simp} \int_{\mathbb{R}^d} f \)

for every \( c \in [0, \infty) \); (unsigned linearity)

(ii) \( \text{Simp} \int_{\mathbb{R}^d} f < \infty \) iff \( f \) is finite almost everywhere and its support has finite measure; (finiteness)

(iii) \( \text{Simp} \int_{\mathbb{R}^d} f = 0 \) iff \( f = 0 \) almost everywhere; (vanishing)

(iv) If \( f \) and \( g \) agree almost everywhere, \( \text{Simp} \int_{\mathbb{R}^d} f = \text{Simp} \int_{\mathbb{R}^d} g \); (equivalence)

(v) If \( f(x) \leq g(x) \) for almost every \( x \in \mathbb{R}^d \), \( \text{Simp} \int_{\mathbb{R}^d} f \leq \text{Simp} \int_{\mathbb{R}^d} g \); (monotonicity)

(vi) For any Lebesgue measurable set \( S \), \( \text{Simp} \int_{\mathbb{R}^d} 1_S = m(S) \). (compatibility)

Definition: A complex-valued simple function \( f : \mathbb{R}^d \to \mathbb{C} \) is said to be absolutely integrable if \( \text{Simp} \int_{\mathbb{R}^d} |f| < \infty \).

Definition: If the real-valued function \( f \) is absolutely integrable, let

\( \text{Simp} \int_{\mathbb{R}^d} f = \text{Simp} \int_{\mathbb{R}^d} f_+ - \text{Simp} \int_{\mathbb{R}^d} f_- \)

in terms of the unsigned simple functions \( f_+ = \max(f, 0) \) and \( f_- = \max(-f, 0) \) (which are both dominated by \( |f| \)).
**Definition:** If a complex-valued simple function $f : \mathbb{R}^d \to \mathbb{C}$ is absolutely integrable,

$$\text{Simp} \int_{\mathbb{R}^d} f = \text{Simp} \int_{\mathbb{R}^d} \text{Re } f + i \text{Simp} \int_{\mathbb{R}^d} \text{Im } f$$

**Remark:** We note that a complex-valued simple function $f$ is absolutely integrable iff its support has finite measure. In particular, the space $\text{Simp}^{\text{abs}}(\mathbb{R}^d)$ of absolutely integrable simple functions, begin closed under addition and scalar multiplication by complex numbers, is a complex vector space.

**Problem 1.20:** (Properties of the complex-valued simple integral). Let $f, g : \mathbb{R}^d \to \mathbb{C}$ be simple complex-valued functions. Then

(i)

$$\text{Simp} \int_{\mathbb{R}^d} (f + g) = \text{Simp} \int_{\mathbb{R}^d} f + \text{Simp} \int_{\mathbb{R}^d} g$$

and

$$\text{Simp} \int_{\mathbb{R}^d} cf = c \text{Simp} \int_{\mathbb{R}^d} f$$

for every $c \in \mathbb{C}$, along with

$$\text{Simp} \int_{\mathbb{R}^d} f = \text{Simp} \int_{\mathbb{R}^d} f;$$

(*-linearity)

(ii) If $f$ and $g$ agree almost everywhere, $\text{Simp} \int_{\mathbb{R}^d} f = \text{Simp} \int_{\mathbb{R}^d} g$; (equivalence)

(iii) For any Lebesgue measurable set $S$, $\text{Simp} \int_{\mathbb{R}^d} 1_S = m(S)$. (compatibility)

Hint: Use the decomposition

$$f + g = (f + g)_+ - (f + g)_- = (f_+ - f_-) + (g_+ - g_-).$$

**Definition:** An unsigned function $f : \mathbb{R}^d \to [0, \infty]$ is unsigned Lebesgue measurable (or measurable) if it is the pointwise limit of a sequence of unsigned simple functions.
**Definition:** Let $X \subset \mathbb{R}^d$. A set $U \subset X$ is relatively open (relatively closed) in $X$ if there is an open (closed) set $V$ in $\mathbb{R}^d$ such that $U = V \cap X$.

**Lemma 1.14** (Characterization of measurable unsigned functions):

Let $f : \mathbb{R}^d \to [0, \infty]$ be an unsigned function. The following are equivalent:

(i) $f$ is unsigned Lebesgue measurable.

(ii) $f$ is the pointwise limit of a sequence of unsigned simple functions.

(iii) $f$ is the pointwise almost everywhere limit of unsigned simple functions.

(iv) $f = \sup_n f_n$ for an increasing sequence $f_n$ of bounded unsigned simple functions that have finite-measure support.

(v) For every $\lambda \in [0, \infty]$, the set $\{ x \in \mathbb{R}^d : f(x) > \lambda \}$ is Lebesgue measurable.

(vi) For every $\lambda \in [0, \infty]$, the set $\{ x \in \mathbb{R}^d : f(x) \geq \lambda \}$ is Lebesgue measurable.

(vii) For every $\lambda \in [0, \infty]$, the set $\{ x \in \mathbb{R}^d : f(x) < \lambda \}$ is Lebesgue measurable.

(viii) For every $\lambda \in [0, \infty]$, the set $\{ x \in \mathbb{R}^d : f(x) \leq \lambda \}$ is Lebesgue measurable.

(ix) For every interval $I \subset [0, \infty)$, the set $f^{-1}(I) = \{ x \in \mathbb{R}^d : f(x) \in I \}$ is Lebesgue measurable.

(x) For every relatively open set $U \subset [0, \infty)$, the set $f^{-1}(U) = \{ x \in \mathbb{R}^d : f(x) \in U \}$ is Lebesgue measurable.

(xi) For every relatively closed set $F \subset [0, \infty)$, the set $f^{-1}(F) = \{ x \in \mathbb{R}^d : f(x) \in F \}$ is Lebesgue measurable.

**Proof:**

(i) $\iff$ (ii) This is the definition of Lebesgue measurability of an unsigned function.

(ii) $\Rightarrow$ (iii) Everywhere implies almost everywhere.

(iv) $\Rightarrow$ (ii) Every monotone sequence in $[0, \infty]$ converges.

(iii) $\Rightarrow$ (v) We are given that for almost all $x \in \mathbb{R}^d$,

$$f(x) = \lim_{n \to \infty} f_n(x) = \limsup_{n \to \infty} f_n(x) = \lim_{n \to \infty} \sup_{k \geq n} f_k(x) = \inf_{n \in \mathbb{N}} \sup_{k \geq n} f_k(x).$$

Then to within a set of measure zero, the set $\{ x \in \mathbb{R}^d : f(x) > \lambda \}$ equals

$$\bigcup_{M \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \left\{ x \in \mathbb{R}^d : \sup_{k \geq n} f_k(x) > \lambda + \frac{1}{M} \right\},$$
or equivalently,

\[
\bigcup_{M \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} \left\{ x \in \mathbb{R}^d : f_k(x) > \lambda + \frac{1}{M} \right\}.
\]

But each set \( \{ x \in \mathbb{R}^d : f_k(x) > \lambda + 1/M \} \) is Lebesgue measurable since the \( f_k \) are unsigned simple functions. Since countable unions and intersections of Lebesgue measurable sets are Lebesgue measurable, we arrive at (v).

(v) \iff (vi)

(vii) \iff (viii) To establish these two equivalences, let \( \mathbb{Q}^+ = \mathbb{Q} \cap [0, \infty) \). Then for \( \lambda \in (0, \infty) \),

\[
\{ x \in \mathbb{R}^d : f(x) \geq \lambda \} = \bigcap_{\lambda' \in \mathbb{Q}^+ : \lambda' < \lambda} \{ x \in \mathbb{R}^d : f(x) > \lambda' \}.
\]

Likewise for \( \lambda \in [0, \infty) \),

\[
\{ x \in \mathbb{R}^d : f(x) > \lambda \} = \bigcup_{\lambda' \in \mathbb{Q}^+ : \lambda' > \lambda} \{ x \in \mathbb{R}^d : f(x) \geq \lambda' \}.
\]

Since \( \mathbb{Q}^+ \) is countable, we then see that (v) and (vi) are equivalent and so are (vii) and (viii).

(v) \iff (viii)

(vi) \iff (vii)

(x) \iff (xi) These three equivalences follow immediately upon taking complements.

(v)–(viii) \implies (ix) Every interval \( I \subset [0, \infty) \) can be expressed as the intersection of two semi-infinite intervals (half lines).

(x) \implies (vii) Let \( U = [0, \lambda) \).

(ix) \implies (x) Every open set in \( [0, \infty) \) is the union of countably many open intervals.

(ix) \implies (iv) For every \( n \in \mathbb{N} \), let \( f_n(x) \) be the largest integer multiple of \( 2^{-n} \) bounded by \( \min(f(x), n) \) for \( x \in B_n[0] \) and zero elsewhere. At each \( x \), we see that \( f(x) \) is the supremum of the increasing sequence of functions \( f_n(x) \). Note that \( f_n \) achieves each of its finite number of nonzero values \( c \) on a Lebesgue measurable set \( f_n^{-1}(c) = f^{-1}(I_c) \cap B_n[0] \), where \( I_c \subset [0, \infty) \) is an interval or half line. Thus each function \( f_n \) is a bounded unsigned simple function with finite-measure support.
Remark: Having established these characterizations of measurable functions, we now observe that many of the unsigned functions that arise in practical applications are measurable:

- every continuous unsigned function \( f : \mathbb{R}^d \to [0, \infty) \);
- every unsigned simple function;
- the supremum, infimum, limit superior, and limit inferior of unsigned measurable functions;
- an unsigned function that is almost everywhere equal to an unsigned measurable function;
- the composition \( \phi \circ f \) of a continuous function \( \phi : [0, \infty] \to [0, \infty] \) and an unsigned measurable function \( f \);
- the sum and product of unsigned measurable functions.

Remark: If an unsigned measurable function \( f \) is bounded by \( M \), the functions \( f_n \) constructed in Lemma 1.14 (ix) \( \Rightarrow \) (iv) are each bounded by \( M \). Thus, \( f \) is a bounded unsigned measurable function iff it is the limit of a uniformly bounded sequence of simple functions.

Problem 1.21: Show that an unsigned function \( f : \mathbb{R}^d \to [0, \infty] \) is a simple function iff it is measurable and takes on finitely many values.

Remark: If \( f \) is measurable, Lemma 1.14 shows that \( f^{-1}(S) \) is Lebesgue measurable for many, but not all, measurable sets \( S \) (for a counterexample, see Tao, Remark 1.3.10).

Definition: An almost everywhere-defined complex-valued function \( f : \mathbb{R}^d \to \mathbb{C} \) is \textit{Lebesgue measurable} (or \textit{measurable}) if it is the pointwise almost-everywhere limit of a sequence of complex-valued simple functions.

Lemma 1.15 (Characterization of measurable complex-valued functions): Let \( f : \mathbb{R}^d \to \mathbb{C} \) be an almost-everywhere defined complex-valued function. The following are equivalent:

(i) \( f \) is measurable.
(ii) \( f \) is the pointwise almost-everywhere limit of a sequence of complex-valued simple functions.

(iii) The magnitudes of the positive and negative parts of \( \text{Re} \, f \) and \( \text{Im} \, f \) are unsigned measurable functions.

(iv) For every open set \( U \subset \mathbb{C} \), the set \( f^{-1}(U) \) is measurable.

(v) For every closed set \( F \subset \mathbb{C} \), the set \( f^{-1}(F) \) is measurable.

Problem 1.22: Let \( f : \mathbb{R}^d \rightarrow \mathbb{C} \). Show that

(i) if \( f \) is continuous, it is measurable;

(ii) \( f \) is simple iff it is measurable and takes on finitely many values;

(iii) if \( f \) is almost everywhere equal to a measurable function, it is itself measurable.

(iv) if a sequence \( f_n \) of complex-valued measurable functions converges pointwise almost everywhere to \( f \), then \( f \) is measurable.

(v) if \( f \) is measurable, the composition \( \phi \circ f \) of a continuous function \( \mathbb{C} \rightarrow \mathbb{C} \) and \( f \) is measurable.

Problem 1.23: Show that the sum and product of measurable functions are measurable.

Definition: Let \( f : \mathbb{R}^d \rightarrow [0, \infty] \) be an unsigned (but not necessarily measurable) function. The lower unsigned Lebesgue integral is

\[
\int_{\mathbb{R}^d} f \overset{\text{def}}{=} \sup_{h \text{ simple}, 0 \leq h \leq f} \text{Simp} \int_{\mathbb{R}^d} h.
\]

Likewise, the upper unsigned Lebesgue integral is

\[
\int_{\mathbb{R}^d} f \overset{\text{def}}{=} \inf_{h \text{ simple}, h \geq f} \text{Simp} \int_{\mathbb{R}^d} h.
\]
**Remark:** For any unsigned function \( f : \mathbb{R}^d \to [0, \infty] \) observe that

\[
\int_{\mathbb{R}^d} f \leq \underint_{\mathbb{R}^d} f
\]

**Theorem 1.3** (Properties of the lower and upper Lebesgue integrals): Let \( f, g : \mathbb{R}^d \to \mathbb{C} \) be unsigned (not necessarily measurable) functions. Then

(i) if \( f \) is simple, \( \int_{\mathbb{R}^d} f = \underint_{\mathbb{R}^d} f = \text{Simp} \int_{\mathbb{R}^d} f \); compatibility

(ii) if \( f \leq g \) pointwise almost everywhere, \( \int_{\mathbb{R}^d} f \leq \int_{\mathbb{R}^d} g \) and \( \overline{\int_{\mathbb{R}^d}} f \leq \overline{\int_{\mathbb{R}^d}} g \); monotonicity

(iii) \( \int_{\mathbb{R}^d} cf = c \int_{\mathbb{R}^d} f \) for every \( c \in [0, \infty) \); scaling

(iv) if \( f \) and \( g \) agree almost everywhere, \( \int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} g \) and \( \overline{\int_{\mathbb{R}^d}} f = \overline{\int_{\mathbb{R}^d}} g \); equivalence

(v) \( \int_{\mathbb{R}^d} (f + g) \geq \int_{\mathbb{R}^d} f + \int_{\mathbb{R}^d} g \); lower superadditivity

(vi) \( \int_{\mathbb{R}^d} (f + g) \leq \int_{\mathbb{R}^d} f + \int_{\mathbb{R}^d} g \); upper subadditivity

(vii) for any measurable set \( S \subset \mathbb{R}^d \),

\[
\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} f 1_S + \int_{\mathbb{R}^d} f 1_{S^c};
\]
complementarity

(viii)

\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} \min(f(x), n) \, dx = \int_{\mathbb{R}^d} f;
\]
vertical truncation

(ix)

\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} f(x) 1_{B_n[0]} \, dx = \int_{\mathbb{R}^d} f;
\]
(use the monotone convergence theorem) horizontal truncation

(x) if \( f + g \) is a bounded simple function with finite measure support,

\[
\text{Simp} \int_{\mathbb{R}^d} (f + g) = \int_{\mathbb{R}^d} f + \int_{\mathbb{R}^d} g.
\]
reflection
Definition: If \( f : \mathbb{R}^d \to [0, \infty] \) is measurable, we define the unsigned Lebesgue integral \( \int_{\mathbb{R}^d} f \) to be the lower lower unsigned Lebesgue integral \( \int_{\mathbb{R}^d} f \).

Problem 1.24: Let \( f : \mathbb{R}^d \to [0, \infty) \) be measurable, bounded, and vanishing outside of a set of finite measure. Show that the lower and upper Lebesgue integrals agree. Hint: use the fact that a bounded unsigned measurable function is the pointwise limit of a uniformly bounded sequence of simple functions.

Corollary 1.3.1 (Finite additivity of the Lebesgue integral): Let \( f, g : \mathbb{R}^d \to [0, \infty] \) be measurable. Then \( \int_{\mathbb{R}^d} (f + g) = \int_{\mathbb{R}^d} f + \int_{\mathbb{R}^d} g \).

Proof: We first prove this in the case where \( f \) and \( g \) are bounded and have bounded support. By Problem 1.24, the lower and upper integrals of \( f, g, \) and \( f + g \) agree. The result then follows from lower superadditivity and upper subadditivity. The general case can then be reduced to this case by applying horizontal and vertical truncation and taking the corresponding limits.

Problem 1.25: Show for an arbitrary set \( S \subset \mathbb{R}^d \) that \( \overline{\int_{\mathbb{R}^d} 1_S} = m^*(S) \).

From outer regularity, we know that
\[
m^*(S) = \inf_{U \supset S, U \text{ open}} m^*(U) = \inf_{U \supset S, U \text{ open}} \int_{\mathbb{R}^d} 1_U \geq \int_{\mathbb{R}^d} 1_S,
\]
since \( 1_U \geq 1_S \). Furthermore, given \( \epsilon > 0 \), there exists a simple function \( h_\epsilon \geq 1_S \) such that \( h_\epsilon \geq 1 \) on a measurable set \( T_\epsilon \supset S \) and
\[
\int_{\mathbb{R}^d} 1_S + \epsilon > \int_{\mathbb{R}^d} h_\epsilon \geq \int_{\mathbb{R}^d} 1_{T_\epsilon} = m(T_\epsilon) \geq m^*(S).
\]
Since \( \epsilon \) is arbitrary, we conclude that \( \overline{\int_{\mathbb{R}^d} 1_S} = m^*(S) \).

Remark: In view of the fact that Lebesgue outer measure is not necessarily additive, a consequence of Problem 1.25 is that the upper and lower Lebesgue integrals need not be additive.

Problem 1.26: If \( f : \mathbb{R}^d \to [0, \infty] \) is Lebesgue measurable, show that the Lebesgue measure of \( \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : 0 \leq y \leq f(x)\} \) exists and equals \( \int_{\mathbb{R}^d} f \).

Remark: The statement in Problem 1.26 can be used as an alternate definition of the Lebesgue integral of a measurable function.
Remark: The Lebesgue integral is the unique map from measurable unsigned functions over $\mathbb{R}^d$ that is compatible with the simple integral and obeys finite additivity, along with the horizontal and vertical truncation properties.

Remark: We can extend a given function $f : [a, b] \rightarrow [0, \infty)$ to $\mathbb{R}$ by assigning it the value 0 on $\mathbb{R} \setminus [a, b]$. If $f$ is Riemann integrable on $[a, b]$, then it is the pointwise limit of a sequence of piecewise constant functions and is therefore measurable. Since a lower sum for $f$ is the integral of a simple function bounded above by $f$, it must be less than or equal to $\int_{\mathbb{R}^d} f$. Likewise, an upper sum for $f$ must be greater than or equal to $\int_{\mathbb{R}^d} f$. Let $L$ be the lower Riemann integral (the supremum of all lower sums) and $U$ be the upper Riemann integral (the infimum of all upper sums). Then $L \leq \int_{\mathbb{R}^d} f \leq \int_{\mathbb{R}^d} f \leq U$. Since $L = U = \int_a^b f$, we conclude that $\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} f = \int_a^b f$. That is, the Lebesgue integral of a Riemann integrable function exists and equals $\int_a^b f$.

Definition: A measurable complex-valued function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is said to be absolutely integrable if the $L^1$ semi-norm

$$|f|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |f| < \infty.$$ 

Definition: The space of all absolutely integrable functions is denoted $L^1(\mathbb{R}^d)$.

Definition: If $f$ is real-valued and absolutely integrable, we define

$$\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} f_+ - \int_{\mathbb{R}^d} f_-,$$

where $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$.

Definition: If $f$ is complex-valued and absolutely integrable, we define

$$\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} \text{Re} f + i \int_{\mathbb{R}^d} \text{Im} f$$

Theorem 1.4 (Markov's inequality): Let $f : \mathbb{R}^d \rightarrow \mathbb{C}$ be measurable. Then for every $\lambda \in (0, \infty)$,

$$m(\{x \in \mathbb{R}^d : |f(x)| \geq \lambda\}) \leq \frac{1}{\lambda} \int_{\mathbb{R}^d} |f|.$$
Proof: In view of the pointwise inequality
\[ \lambda 1_{\{x \in \mathbb{R}^d : |f(x)| \geq \lambda \}} \leq |f(x)|, \]
we see from the definition of the lower Lebesgue integral that
\[ \lambda m(\{x \in \mathbb{R}^d : |f(x)| \geq \lambda \}) \leq \int_{\mathbb{R}^d} |f| \).

**Corollary 1.4.1:** Let \( f : \mathbb{R}^d \to \mathbb{C} \) be measurable.

(i) If \( \int_{\mathbb{R}^d} |f| < \infty \), then \( f \) is finite almost everywhere.

(ii) \( \int_{\mathbb{R}^d} |f| = 0 \) iff \( f \) is zero almost everywhere.

Proof:

(i) Consider the limit \( \lambda \to \infty \).

(ii) Consider the limit \( \lambda \to 0 \).

**Remark:** The converse to Corollary 1.4.1 (i) is false: consider \( \int_{\mathbb{R}^d} 1 = m(\mathbb{R}^d) = \infty \).

**Remark:** The use of the integral \( \int_{\mathbb{R}^d} |f| \) to control the distribution of \( f \) is the first moment method. In probability theory, one also uses higher moments such as \( \int_{\mathbb{R}^d} |f|^p \) and Fourier moments \( \int_{\mathbb{R}^d} e^{itf} \) to control the distribution of \( f \).

**Remark:** From the pointwise triangle inequality \( |f(x) + g(x)| \leq |f(x)| + |g(x)| \), we obtain the \( L^1 \) triangle inequality \( |f + g|_{L^1(\mathbb{R}^d)} \leq |f|_{L^1(\mathbb{R}^d)} + |g|_{L^1(\mathbb{R}^d)} \) for every measurable function \( f, g : \mathbb{R}^d \to \mathbb{C} \). Note also for \( c \in \mathbb{C} \) that \( |cf|_{L^1(\mathbb{R}^d)} = |c||f|_{L^1(\mathbb{R}^d)} \) and that \( |f|_{L^1(\mathbb{R}^d)} = 0 \) iff \( f \) vanishes almost everywhere. Together these properties make \( L^1(\mathbb{R}^d) \) a complex vector space, with semi-norm \( L^1 \) (it is a semi-norm because of the almost everywhere qualifier).

**Remark:** The Lebesgue integral is a *-linear operator from \( L^1(\mathbb{R}^d) \) to \( \mathbb{C} \).

**Problem 1.27:** (Translation invariance)
If \( f \in L^1(\mathbb{R}^d) \), show for every \( y \in \mathbb{R} \) that \( \int_{\mathbb{R}^d} f(x + y) \, dx = \int_{\mathbb{R}^d} f(x) \, dx \).

**Problem 1.28:** (Linear change of variables)
If \( f \in L^1(\mathbb{R}^d) \) and \( T \) is an invertible linear transformation, show that
\[ \int_{\mathbb{R}^d} f(T^{-1}x) \, dx = |\det T| \int_{\mathbb{R}^d} f. \]
Problem 1.29: If $S$ and $T$ are disjoint measurable subsets of $\mathbb{R}^d$ and $f : S \cup T \to \mathbb{C}$ is absolutely integrable, show that
\[
\int_{S \cup T} (f1_S) = \int_S f
\]
and
\[
\int_S f + \int_T f = \int_{S \cup T} f.
\]

Lemma 1.16 (Triangle inequality): Let $f \in L^1(\mathbb{R}^d)$. Then
\[
\left| \int_{\mathbb{R}^d} f \right| \leq \int_{\mathbb{R}^d} |f|.
\]

Proof: If $f = f_+ - f_-$ is real-valued then $|f| = f_+ + f_-$ and the claim follows from the triangle inequality for $\mathbb{R}$. If $f$ is complex-valued we can express
\[
\left| \int_{\mathbb{R}^d} f \right| = e^{i\theta} \int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} e^{i\theta} f
\]
for some real phase $\theta$. On taking real parts, we find
\[
\left| \int_{\mathbb{R}^d} f \right| = \int_{\mathbb{R}^d} \text{Re} (e^{i\theta} f) \leq \int_{\mathbb{R}^d} |e^{i\theta} f| = \int_{\mathbb{R}^d} |f|.
\]

Definition: A step function is a finite linear combination of indicator functions $1_B$ of boxes $B$.

Theorem 1.5 (Approximation of $L^1$ functions): Let $f \in L^1(\mathbb{R}^d)$ and $\epsilon > 0$. There exists

(i) an absolutely integrable simple function $g$ such that $|f - g|_{L^1(\mathbb{R}^d)} < \epsilon$.

(ii) a step $g$ such that $|f - g|_{L^1(\mathbb{R}^d)} < \epsilon$.

(iii) a continuous, compactly supported function $g \in L^1(\mathbb{R}^d)$ such that $|f - g|_{L^1(\mathbb{R}^d)} < \epsilon$.

Proof:

(i) In the case where $f$ is unsigned, by the definition of the lower Lebesgue integral there exists an unsigned simple function $g \leq f$ such that $\int_{\mathbb{R}^d} g > \int_{\mathbb{R}^d} f - \epsilon$, which implies that $|f - g|_{L^1(\mathbb{R}^d)} < \epsilon$. This result can then immediately be generalized to the real-valued and complex-valued case.
(ii) The case where $f$ is a simple function can be reduced, using linearity and the triangle inequality, to the case where $f$ is the indicator function of a set with finite measure. The claim then follows from the fact that such sets can be approximated by an elementary set. The general case then follows on applying (i) and the triangle inequality.

(iii) It suffices to show this in the case where $f$ is the indicator function of a box $B$. Let $B'$ be a slightly enlarged box that contains $B$ within its interior, such that $|B'| < |B| + \epsilon$. Let $g(x) = \max(1 - C \operatorname{dist}(x, B), 0)$ where $C$ is chosen sufficiently large such that $g(x) = 0$ outside of $B'$. Then $g$ is continuous and compactly supported, with $\|f - g\|_{L^1(\mathbb{R}^d)} < \epsilon$.

**Definition:** A sequence of functions $f_n : \mathbb{R}^d \to \mathbb{C}$ converges *locally uniformly* to a $f : \mathbb{R}^d \to \mathbb{C}$ if for every bounded subset $S \subset \mathbb{R}^d$, $f_n : \mathbb{R}^d \to C$ converges uniformly to $f$ on $S$.

- The sequence of functions $x \mapsto x/n$ on $\mathbb{R}$ for $n = 1, 2, \ldots$ converges locally uniformly (and hence pointwise) to 0 on $\mathbb{R}$, but not uniformly.

- The partial sums $\sum_{k=0}^n x^k/k!$ of the Taylor series of $e^x$ converges to $e^x$ locally uniformly on $\mathbb{R}$, but not uniformly.

- The functions
  $$f_n(x) = \begin{cases} 1/nx & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$
  converge pointwise everywhere to zero as $n \to \infty$, but not locally uniformly (due to the behaviour of $f_n$ near $x = 0$).

**Remark:** Although pointwise convergence is evidently a weaker mode of convergence than local uniform convergence, the following theorem establishes that one can recover local uniform convergence if one is willing to delete as set of arbitrarily small measure.

**Theorem 1.6** (Egorov’s theorem): Let $f_n : \mathbb{R}^d \to \mathbb{C}$ be a sequence of measurable functions that converge pointwise almost everywhere to $f : \mathbb{R}^d \to \mathbb{C}$. Given $\epsilon > 0$, there exists a Lebesgue measurable set $S$ of measure at most $\epsilon$ such that $f_n$ converges locally uniformly to $f$ outside of $S$. 
Proof: By modifying $f_n$ and $f$ as needed on a null set (which can be absorbed into $S$), we may assume that $f_n$ converges pointwise to $f$ on $\mathbb{R}^d$. That is, for each $x \in \mathbb{R}^d$ and $m \in \mathbb{N}$, the Lebesgue measurable set

$$S_N = \{ x \in \mathbb{R}^d : |f_n(x) - f(x)| \geq 1/m \text{ for some } n > N \}$$

obeys

$$\bigcap_{N=1}^{\infty} S_N = \emptyset.$$

On exploiting the downward monotone convergence of the decreasing sequence of sets $S_N$, we find

$$\lim_{N \to \infty} m(S_N \cap B_m(0)) = 0.$$

In particular, for each $m \in \mathbb{N}$, we can find $N_m \in \mathbb{N}$ such that

$$N > N_m \Rightarrow m(S_N \cap B_m(0)) < \frac{\epsilon}{2m}.$$

Then by countable subadditivity, the Lebesgue measurable set

$$S = \bigcup_{m=1}^{\infty} S_N \cap B_m(0)$$

has measure less than $\epsilon$. We thus see that for every $m \geq 1$ and $n > N_m$ that $|f_n(x) - f(x)| < 1/m$ ($f_n$ converges uniformly to $f$) on $x \in B_m(0) \setminus S$. Since every bounded set is contained within a ball $B_m(0)$ for some $m$, the claim follows.

**Remark:** We have now witnessed the three heuristic principles of measure theory first articulated by Littlewood:

1. Every measurable set is nearly a finite union of boxes;
2. Every absolutely integrable function is nearly continuous;
3. Every pointwise convergent sequence of functions is nearly locally uniformly convergent.

### 1.F Abstract Measure Spaces

**Definition:** Let $X$ be a set. A *Boolean algebra* on $X$ is a collection $\mathcal{B}$ of subsets of $X$ such that:

(i) $\emptyset \in \mathcal{B}$; (empty set)

(ii) If $S \in \mathcal{B}$, then the complement $S^c = X \setminus S$ is also an element of $\mathcal{B}$; (closure under complement)

(iii) If $S, T \in \mathcal{B}$, then $S \cup T \in \mathcal{B}$; (closure under finite union)
**Definition:** Given two Boolean algebras $\mathcal{B}$, $\mathcal{B}'$ on $X$, we say that $\mathcal{B}$ is finer than (coarser than) $\mathcal{B}'$ if $\mathcal{B} \supset \mathcal{B}'$ ($\mathcal{B} \subset \mathcal{B}'$).

- The coarsest Boolean algebra on a set $X$ is the trivial algebra $\{\emptyset, X\}$.
- The finest Boolean algebra on a set $X$ is the discrete algebra $\mathcal{P}(X) = \{S : S \subset X\}$.

**Remark:** All other Boolean algebras are intermediate between these two extremes: finer than the trivial algebra, but coarser than the discrete one.

- The elementary Boolean algebra on $\mathbb{R}^d$ is the collection of subsets of $\mathbb{R}^d$ that are either elementary or have an elementary complement.
- The Jordan algebra on $\mathbb{R}^d$ is the collection of subsets of $\mathbb{R}^d$ that are either Jordan-measurable or have a Jordan-measurable complement.
- The Lebesgue algebra $\mathcal{L}[\mathbb{R}^d]$ on $\mathbb{R}^d$ is the collection of Lebesgue-measurable subsets of $\mathbb{R}^d$.

**Remark:** The Lebesgue algebra is finer than the Jordan algebra, which is itself finer than the elementary Boolean algebra.

- The null algebra is the collection of sets in $\mathbb{R}^d$ that are either Lebesgue null sets or have null complements.

**Remark:** The null algebra is coarser than the Lebesgue algebra.

**Remark:** Let $\mathcal{F}$ be a family of subsets of $X$. The intersection $\langle \mathcal{F} \rangle_{\text{bool}}$ of all Boolean algebras that contain $\mathcal{F}$ is itself a Boolean algebra. It is the coarsest Boolean algebra that contains $\mathcal{F}$; we say that $\langle \mathcal{F} \rangle_{\text{bool}}$ is generated by $\mathcal{F}$.

**Definition:** Suppose we express a set $X$ as a union $\bigcup_{\alpha \in I} A_\alpha$ of disjoint sets $A_\alpha$, called atoms, where $I$ is an index set. This partitioning of $X$ generates a Boolean algebra, the atomic algebra $\mathcal{A}(\{A_\alpha : \alpha \in I\})$, defined as the collection of all unions $\bigcup_{\alpha \in J} A_\alpha$ such that $J \subset I$.

**Remark:** The trivial algebra corresponds to the trivial partition of $X$ into a single atom, namely $X$ itself.
Remark: The discrete algebra corresponds to the discrete partition of $X = \bigcup_{x \in X} \{x\}$ into singleton atoms.

- Let $n$ be an integer. The dyadic algebra $D_n(\mathbb{R}^d)$ at scale $2^{-n}$ is the atomic algebra generated by taking unions and complements of half-open dyadic cubes
  \[
  \left[\frac{i_1}{2^n}, \frac{i_1 + 1}{2^n}\right) \times \ldots \times \left[\frac{i_d}{2^n}, \frac{i_d + 1}{2^n}\right)
  \]
  for integers $i_1, i_2, \ldots, i_d$. Note that $D_{n+1} \supset D_n$.

Definition: A Boolean algebra is finite if it contains only finitely many sets.

Remark: Every finite Boolean algebra is an atomic algebra.

Remark: The elementary, Jordan, Lebesgue, and null algebras are not atomic algebras: they are not composed of indivisible atoms.

Definition: Let $X$ be a set. A $\sigma$-algebra on $X$ is a collection $\mathcal{B}$ of $X$ such that

(i) $\emptyset \in \mathcal{B}$; \hspace{1cm} (empty set)

(ii) If $S \in \mathcal{B}$, then the complement $S^c = X \setminus S$ is also an element of $\mathcal{B}$; \hspace{1cm} (closure under complement)

(iii) If $S_1, S_2, \ldots \in \mathcal{B}$, then $\bigcup_{n=1}^{\infty} S_n \in \mathcal{B}$. \hspace{1cm} (closure under countable union)

- All atomic algebras are $\sigma$-algebras.

- The Lebesgue and null algebras are $\sigma$-algebras, but the elementary and Jordan algebras are not.

- Every $\sigma$-algebra is a Boolean algebra.

- Every finite Boolean algebra is an atomic algebra.

Remark: An intersection $\bigcap_{\alpha \in I} \mathcal{B}_\alpha$ of $\sigma$-algebras $\mathcal{B}_\alpha$ is itself a $\sigma$-algebra and is the finest $\sigma$-algebra that is coarser than each of the $\mathcal{B}_\alpha$. 
Remark: Let $\mathcal{F}$ be a family of subsets of $X$. The intersection $\langle \mathcal{F} \rangle$ of all $\sigma$-algebras that contain $\mathcal{F}$ is itself a $\sigma$-algebra. It is the coarsest $\sigma$-algebra that contains $\mathcal{F}$; we say that $\langle \mathcal{F} \rangle$ is generated by $\mathcal{F}$.

Remark: Observe that $\langle \mathcal{F} \rangle_{\text{bool}} \subset \langle \mathcal{F} \rangle$, with equality holding iff $\langle \mathcal{F} \rangle_{\text{bool}}$ is a $\sigma$-algebra.

- Let $\mathcal{F}$ be the collection of all boxes in $\mathbb{R}^d$. Then $\langle \mathcal{F} \rangle_{\text{bool}}$ is the elementary algebra, which is not a $\sigma$-algebra.

Definition: Let $X$ be a set and $\mathcal{B}$ be a $\sigma$-algebra. We refer to the pair $(X, \mathcal{B})$ as a measurable space.

Remark: In abstract measure theory, the $\sigma$-algebra $\mathcal{B}$ identifies the subsets of $X$ that one is allowed to measure.

Definition: Let $X$ be a metric space. The Borel $\sigma$-algebra $\mathcal{B}[X]$ on $X$ is the $\sigma$-algebra generated by the collection of open subsets of $X$. The elements of $\mathcal{B}[X]$ are Borel measurable.

- The Borel $\sigma$-algebra contains all open sets, all closed sets, all $G_\delta$ sets, and all $F_\sigma$ sets (along with countable unions and intersections thereof).

Remark: Since every open set in $\mathbb{R}^d$ is Lebesgue measurable, the Borel $\sigma$-algebra is coarser than the Lebesgue $\sigma$-algebra.

Remark: Let $\mathcal{F}$ be a family of subsets of $X$ with cardinality $\kappa$. Using transfinite induction, one can show that $\langle \mathcal{F} \rangle$ has cardinality at most $\kappa^{\aleph_0}$, where $\aleph_0$ denotes the cardinality of $\mathbb{N}$.

Remark: Since every open set in $\mathbb{R}^d$ can be expressed as a countable union of open balls (centered on a rational $d$-tuple, with rational radius), the cardinality of the generator of open sets is the same as the cardinality $\aleph_0$ of the rationals. Then $\mathcal{B}[\mathbb{R}^d]$ has cardinality at most $\aleph_0^{\aleph_0}$, which is the same as the cardinality $c = 2^{\aleph_0}$ of the reals.

Remark: The Cantor set has Lebesgue measure zero, but cardinality $c$. Since any subset of a Lebesgue null set is also a null set, we see that the power set of the Cantor set has cardinality $2^c > c$. Thus, there exist Lebesgue-measurable sets that are not Borel measurable!
Remark: The Lebesgue $\sigma$-algebra on $\mathbb{R}^d$ is generated by the union of the Borel $\sigma$-algebra and the null $\sigma$-algebra.

Definition: Let $\mathcal{B}$ be a Boolean algebra on a set $X$. A finitely additive measure $\mu$ on $\mathcal{B}$ is a map $\mu: \mathcal{B} \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$; \hspace{1cm} \text{nullity}

(ii) If $S$ and $T$ are disjoint, $\mu(S \cup T) = \mu(S) + \mu(T)$; \hspace{1cm} \text{finite additivity}

- The Lebesgue measure $m$ is a finitely additive measure on the Lebesgue $\sigma$-algebra (and hence on the null, Jordan, and elementary sub-algebras).

- Lebesgue outer measure is not finitely additive on the discrete algebra.

- Jordan outer measure is not finitely additive on the Lebesgue algebra.

- Let $x$ be an element of a set $X$ and $\mathcal{B}$ be a Boolean algebra on $X$. The Dirac measure $\delta_x$ at $x$ defined by $\delta_x(S) = 1_S(x)$ is finitely additive.

- The zero measure $0: S \to 0$ is a finitely additive measure on any Boolean algebra.

Remark: A linear combination of finitely additive measures is also a finitely additive measure.

Remark: Let $X$ be a set and $\mathcal{B}$ be a Boolean algebra on $X$. The counting measure $\# : \mathcal{B} \to [0, \infty]$, defined as the cardinality of a finite set and infinity for an infinite set, is a finitely additive measure.

Problem 1.30 (Properties of finitely additive measures): Let $\mu$ be a finitely additive measure on a Boolean $\sigma$-algebra $\mathcal{B}$. Let $S$ and $T$ be $\mathcal{B}$-measurable sets. Show that

(i) If $S \subset T$, then $\mu(S) \leq \mu(T)$; \hspace{1cm} \text{monotonicity}

(ii) $\mu(S \cup T) \leq \mu(S) + \mu(T)$; \hspace{1cm} \text{finite subadditivity}

(iii) If $S$ and $T$ are disjoint, $\mu(S \cup T) = \mu(S) + \mu(T)$; \hspace{1cm} \text{finite additivity}

(iv) $\mu(S \cup T) + \mu(S \cap T) = \mu(S) + \mu(T)$; \hspace{1cm} \text{inclusion–exclusion}

(iv) $\mu(S \cap T) + \mu(S \cup T) = \mu(S \cap T) + \mu(S \setminus T) + \mu(T) = \mu(S) + \mu(T)$. 

Problem 1.31: Let $\mathcal{B}$ be a finite Boolean algebra generated by a finite family $A_1, A_2, \ldots, A_n$ of non-empty atoms. For every finitely additive measure $\mu$ on $\mathcal{B}$, show that there exists unique values $c_1, \ldots, c_n \in [0, \infty]$ such that

$$\mu(S) = \sum_{\substack{1 \leq k \leq n \\ A_k \subseteq S}} c_k \quad \forall S \in \mathcal{B}.$$ 

Equivalently, if $x_k \in A_k$ for $k \in \{1, \ldots, n\}$,

$$\mu = \sum_{k=1}^n c_k \delta_{x_k}.$$ 

Since $\mathcal{B}$ is a finite atomic Boolean algebra, we can express each $S \in \mathcal{B}$ as $S = \bigcup_{1 \leq k \leq n} A_k$. Then $\mu(S) = \sum_{\substack{1 \leq k \leq n \\ A_k \subseteq S}} c_k$, where $c_k = \mu(A_k)$.

Definition: Let $\mathcal{B}$ be a $\sigma$-algebra on a set $X$. A **countably additive measure** or measure $\mu$ on $\mathcal{B}$ is a map $\mu : \mathcal{B} \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$; nullity

(ii) if $S_1, S_2, \ldots$ are disjoint elements of $\mathcal{B}$, then $\mu(\bigcup_{k=1}^\infty S_k) = \sum_{k=1}^\infty \mu(S_k)$. countable additivity

Definition: Let $X$ be a general space, $\mathcal{B}$ be a $\sigma$-algebra, and $\mu(S) \in [0, \infty]$ the measure assigned to each $S \in \mathcal{B}$. We refer to the triple $(X, \mathcal{B}, \mu)$ as a measure space.

- The Lebesgue measure $m$ is a countably additive measure on the Lebesgue $\sigma$-algebra (and hence on every sub-algebra, including the Borel $\sigma$-algebra).

- The Dirac measure is countably additive.

- The counting measure is countably additive.

- The restriction of a countably additive measure to a measurable subspace is again countably additive.
1.F. ABSTRACT MEASURE SPACES

Problem 1.32 (Countable combinations of measures): Let $(X, \mathcal{B})$ be a measurable space.

(i) If $\mu$ is a countably additive measure on $\mathcal{B}$ and $c \in [0, \infty]$, then $c\mu$ is also countably additive on $\mathcal{B}$.

(ii) If $\mu_1, \mu_2, \ldots$ are a sequence of countably additive measures on $\mathcal{B}$, then their sum $\sum_{k=1}^{\infty} \mu_n$ is also countably additive on $\mathcal{B}$.

Remark: Since countably additive measures are also finitely additive, they inherit the monotonicity, finite subadditivity, and inclusion-exclusion properties. In addition, one has further properties.

Problem 1.33: Let $(X, \mathcal{B}, \mu)$ be a measure space. Establish the following properties.

(i) If $S_1, S_2, \ldots$ are $\mathcal{B}$-measurable, then

$$\mu\left(\bigcup_{k=1}^{\infty} S_k\right) \leq \sum_{k=1}^{\infty} \mu(S_k).$$

(countable subadditivity)

(ii) If $S_1 \subset S_2 \subset \ldots$ is an increasing sequence of $\mathcal{B}$-measurable sets, then

$$\mu\left(\bigcup_{k=1}^{\infty} S_k\right) = \lim_{n \to \infty} \mu(S_n) = \sup_n \mu(S_n).$$

(upward monotone convergence)

(iii) If $S_1 \supset S_2 \supset \ldots$ is a decreasing sequence of $\mathcal{B}$-measurable sets and at least one of the $\mu(S_k)$ is finite, then

$$\mu\left(\bigcap_{k=1}^{\infty} S_k\right) = \lim_{n \to \infty} \mu(S_n) = \inf_n \mu(S_n).$$

(downward monotone convergence)

Problem 1.34: (Dominated convergence theorem) Let $(X, \mathcal{B}, \mu)$ be a measure space. Suppose $S_n, n = 1, 2, \ldots$ are $\mathcal{B}$-measurable sets that converge to a set $S$.

(i) Show that $S$ is $\mathcal{B}$-measurable.

(ii) Suppose that the $S_n$ are all contained in another $\mathcal{B}$-measurable set $F$ of finite measure. Show that $m(S_n)$ converges to $m(S)$.

Hint: apply downward monotone convergence to the sets $\bigcap_{n=k}^{\infty} S_n$.

(iii) Give a counterexample to show that the dominated convergence theorem fails if the $S_n$ are not contained in a set of finite measure.
Problem 1.35: Let $X$ be an at most countable set and $\mathcal{B}$ be the discrete $\sigma$-algebra. Show that every measure $\mu$ on $(X, \mathcal{B})$ can be uniquely represented as

$$
\mu(S) = \sum_{x \in S} c_x \quad \forall S \subset X.
$$

for some $c_x \in [0, \infty]$. Equivalently

$$
\mu = \sum_{x \in X} c_x \delta_x.
$$

Definition: A null set of a measure space $(X, \mathcal{B}, \mu)$ is a $\mathcal{B}$-measurable space of measure zero.

Definition: A sub-null set is any subset of a null set.

Definition: A measure space is complete if every sub-null set is a null set.

- The Lebesgue measure space $(\mathbb{R}^d, \mathcal{L}[\mathbb{R}^d], m)$ is complete.

- The Borel measure space $(\mathbb{R}^d, \mathcal{B}[\mathbb{R}^d], m)$ is not complete.

Definition: The completion of a measure space $(X, \mathcal{B}, \mu)$ is its (unique) coarsest refinement $(X, \tilde{\mathcal{B}}, \tilde{\mu})$ that is complete, consisting of sets that differ from a $\mathcal{B}$-measurable set by a $\mathcal{B}$-subnull set.

- The completion of the Borel measure space $(\mathbb{R}^d, \mathcal{B}[\mathbb{R}^d], m)$ is the Lebesgue measure space $(\mathbb{R}^d, \mathcal{L}[\mathbb{R}^d], m)$.

Remark: Recall that a function is continuous if the inverse image of every open set is open. In a similar spirit, in view of Lemma 1.14, we can now generalize the notion of a Lebesgue measurable function:

Definition: Let $(X, \mathcal{B})$ be a measurable space, and let $f : X \to [0, \infty]$ (or $f : X \to \mathbb{C}$) be an unsigned or complex-valued function. We say that $f$ is measurable if $f^{-1}(U)$ is $\mathcal{B}$-measurable for every open subset $U$ of $[0, \infty]$ (or $\mathbb{C}$).
1.F. ABSTRACT MEASURE SPACES

Problem 1.36: Let $(X, \mathcal{B})$ be a measurable space. Show that

(i) a function $f : X \to [0, \infty]$ is measurable iff the level sets $\{x \in X : f(x) > \lambda\}$ are measurable;

(ii) an indicator function $1_S$ of a set $S \subset X$ is measurable iff $S$ is measurable;

(iii) a function $f : X \to [0, \infty]$ (or $f : X \to \mathbb{C}$) is measurable iff $f^{-1}(S)$ is measurable for every Borel-measurable subset $S$ of $[0, \infty]$ (or $\mathbb{C}$);

(iv) a function $f : X \to \mathbb{C}$ is measurable iff its real and imaginary parts are measurable;

(v) a function $f : X \to \mathbb{R}$ is measurable iff the magnitudes $f_+ \doteq \max(f, 0)$, $f_- \doteq \max(-f, 0)$ of its positive and negative parts are measurable;

(vi) the pointwise limit $f$ of a sequence of measurable functions $f_n : X \to [0, \infty]$ (or $\mathbb{C}$) is also measurable;

(vii) if $f : X \to [0, \infty]$ (or $\mathbb{C}$) is measurable and $\phi : [0, \infty] \to [0, \infty]$ (or $\mathbb{C} \to \mathbb{C}$) is continuous, then $\phi \circ f$ is measurable;

(viii) the sum or product of two measurable functions in $[0, \infty]$ (or $\mathbb{C}$) is measurable.

Remark: Recall that the atomic algebra $\mathcal{A}(\{A_\alpha : \alpha \in I\})$ is the collection of all subsets of $X$ that can be represented as the union of one more disjoint atoms $A_\alpha$.

- Every finite Boolean algebra is an atomic algebra.

Problem 1.37: Let $(X, \mathcal{B})$ be a measurable space that is atomic, so that $\mathcal{B} = \mathcal{A}(\{A_\alpha : \alpha \in I\})$ for some partition $\bigcup_{\alpha \in I} A_\alpha$ of $X$ into disjoint nonempty atoms. Show that a function $f : X \to [0, \infty]$ or $f : X \to \mathbb{C}$ is measurable iff it is constant on each atom:

$$f = \sum_{\alpha \in I} c_\alpha 1_{A_\alpha}$$

for some constants $c_\alpha$ in $[0, \infty]$ or in $\mathbb{C}$, as appropriate. Furthermore, $c_\alpha$ are uniquely determined by $f$.

“\(\Leftarrow\)” A linear combination of indicator functions of measurable sets is measurable.

“\(\Rightarrow\)” Since $f^{-1}((0, \lambda))$ and $f^{-1}((\lambda, \infty))$ are both measurable sets, so is $f^{-1}(\lambda) = f^{-1}((\lambda, \infty] \cap [0, \lambda])$ for each constant $\lambda \in [0, \infty]$. Let $A$ be any nonempty atom of $X$ and choose $x \in A$. Then since $f^{-1}(f(x))$ is a measurable set containing $x$, and the smallest such measurable set is $A$, we see that $A \subset f^{-1}(x)$. That is, $f$ is constant on each atom.
Remark: We also have an abstract version of Egorov’s theorem:

**Theorem 1.7** (Egorov’s theorem): Let \((X, \mathcal{B}, \mu)\) be a finite measure space, that is, \(\mu(X) < \infty\) and let \(f_n : X \to \mathbb{C}\) be a sequence of measurable functions that converge pointwise almost everywhere to \(f : X \to \mathbb{C}\). Given \(\epsilon > 0\), there exists a \(\mathcal{B}\)-measurable set \(S\) of measure at most \(\epsilon\) such that \(f_n\) converges locally uniformly to \(f\) outside of \(S\).

**Definition:** Let \((X, \mathcal{B}, \mu)\) be a measure space, with \(\mathcal{B}\) finite (and hence atomic). Let \(\mathcal{B} = \mathcal{A}(\{A_\alpha : \alpha \in I\})\) for some partition \(\bigcup_{\alpha \in I} A_\alpha\) of \(X\) into disjoint nonempty atoms. If \(f : X \to [0, \infty]\) is measurable, it has a unique representation of the form

\[
f = \sum_{k=1}^{n} c_k 1_{A_k}
\]

for some constants \(c_k\) in \([0, \infty]\). We then define the simple integral

\[
\text{Simp} \int_X f \, du \doteq \sum_{k=1}^{n} c_k \mu(A_k)
\]

**Remark:** The precise decomposition of \(\mathcal{B}\) into atoms does not affect the value of the simple integral.

**Remark:** Having defined the simple integral of unsigned measurable functions, we can then construct the simple integral of real-valued and complex-valued valued functions as we did for Lebesgue measurable functions.

**Remark:** We immediately see that we have the monotonicity property \(f \leq g\) implies \(\text{Simp} f \, d\mu \leq \text{Simp} g \, d\mu\) as well as linearity:

\[
\text{Simp} \int_X (f + g) \, d\mu = \text{Simp} \int_X f \, d\mu + \text{Simp} \int_X g \, d\mu
\]

and

\[
\text{Simp} \int_X cf \, d\mu = c \text{Simp} \int_X f \, d\mu
\]

for every measurable functions \(f\) and \(g\), with \(c \in [0, \infty]\).

**Remark:** Let \((X, \mathcal{B}, \mu)\) be a measure space, and \((X, \mathcal{B}, \mu')\) be a refinement of \((X, \mathcal{B}, \mu)\), in the sense that \(\mathcal{B}'\) contains \(\mathcal{B}\) and \(\mu'\) agrees with \(\mu\) on \(\mathcal{B}\). If \(\mathcal{B}'\) is finite, and \(f : \mathcal{B} \to [0, \infty]\) is measurable, then

\[
\text{Simp} \int_X f \, d\mu = \text{Simp} \int_X f \, d\mu'
\]

This allows us to extend the simple integral to simple functions:
**Definition:** An unsigned simple function \( f : X \to [0, \infty] \) on a measurable space \((X, \mathcal{B})\) is a measurable function that takes on finitely many values \(a_1, \ldots, a_k\).

**Definition:** Simple functions are automatically measurable with respect to at least one finite sub-algebra \(\mathcal{B}'\) of \(\mathcal{B}\), namely the \(\sigma\)-algebra \(\mathcal{B}'\) generated by the preimages \(f^{-1}(\{a_1\}), \ldots, f^{-1}(\{a_k\})\). We then define

\[
\text{Simp} \int_X f \, d\mu = \text{Simp} \int_X f \, d\mu|_{\mathcal{B}'},
\]

where \(\mu|_{\mathcal{B}'}\) is the restriction of \(\mu\) to \(\mathcal{B}'\).

**Theorem 1.8** (Properties of the simple integral): Let \((X, \mathcal{B}, \mu)\) be a measure space and let \(f, g : X \to [0, \infty]\) be simple functions. Then

(i) if \(f \leq g\) pointwise, \(\text{Simp} \int_X f \, d\mu \leq \text{Simp} \int_X g \, d\mu\); monotonicity

(ii) \(\text{Simp} \int_X 1_S \, d\mu = \mu(S)\) for every \(\mathcal{B}\)-measurable set \(S\); compatibility

(iii) \(\text{Simp} \int_X cf \, d\mu = c \text{Simp} \int_X f \, d\mu\) for every \(c \in [0, \infty]\); homogeneity

(iv) \(\text{Simp} \int_X (f + g) \, d\mu = \text{Simp} \int_X f \, d\mu + \text{Simp} \int_X g \, d\mu\); finite additivity

(v) if \((X, \mathcal{B}', \mu')\) is a refinement of \((X, \mathcal{B}, \mu)\), \(\text{Simp} \int_X f \, d\mu = \text{Simp} \int_X f \, d\mu'\); refinement

(vi) if \(f(x) = g(x)\) for \(\mu\)-almost every \(x \in X\), \(\text{Simp} \int_X f \, d\mu = \text{Simp} \int_X g \, d\mu\); equivalence

(vii) \(\text{Simp} \int_X f \, d\mu < \infty\) iff \(f\) is finite almost everywhere and is supported on a set of finite measure; finiteness

(viii) \(\text{Simp} \int_X f \, d\mu = 0\) iff \(f\) is zero almost everywhere. vanishing

**Definition:** Let \((X, \mathcal{B}, \mu)\) be a measure space and let \(f : X \to [0, \infty]\) be measurable. The unsigned integral is

\[
\int_X f \, d\mu = \sup_{h \text{ simple}} \text{Simp} \int_X h \, d\mu.
\]
Remark: If $f$ is Lebesgue measurable, this definition reduces to the unsigned Lebesgue integral: $\int_X f \, dm = \int_{\mathbb{R}^d} f$.

**Theorem 1.9** (Properties of the unsigned integral): Let $(X, \mathcal{B}, \mu)$ be a measure space and let $f, g : X \to [0, \infty]$ be measurable. Then

(i) If $f = g \mu$-almost everywhere, then $\int_X f \, d\mu = \int_X g \, d\mu$; equivalence

(ii) if $f \leq g$ for $\mu$-almost everywhere, $\int_X f \, d\mu \leq \int_X g \, d\mu$; monotonicity

(iii) for every $c \in [0, \infty]$, $\int_X cf \, d\mu = c \int_X f \, d\mu$; homogeneity

(iv) $\int_X (f + g) \geq \int_X f + \int_X g$; superadditivity

(v) if $f$ is simple, then $\int_X f \, d\mu = \text{Simp} \int_X f \, d\mu$; compatibility

(vi) for every $\lambda \in (0, \infty)$,

$$\mu(\{x \in \mathbb{R}^d : f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_X f \, d\mu;$$

Markov’s inequality

(vii) if $\int_X f \, d\mu < \infty$, then $f$ is finite for $\mu$-almost every $x$; finiteness

(viii) if $\int_X f \, d\mu = 0$, then $f$ is zero for $\mu$-almost every $x$; vanishing

(ix) $\lim_{n \to \infty} \int_X \min(f, n) \, d\mu = \int_X f \, d\mu$; vertical truncation

(x) if $S_1 \subset S_2 \subset \ldots$ is an increasing sequence of $\mathcal{B}$-measurable sets,

$$\lim_{n \to \infty} \int_X f 1_{S_n} \, d\mu = \int_X f 1_{\bigcup_{n=1}^{\infty} S_n} \, d\mu;$$

horizontal truncation

(xi) If $Y$ is a measurable subset of $X$, then $\int_X f 1_Y \, d\mu = \int_Y f|_Y d\mu|_Y$, where $f|_Y$ and $\mu|_Y$ denote the restriction of $f$ and $\mu$ to $Y$; restriction

**Theorem 1.10** (Properties of the unsigned integral): Let $(X, \mathcal{B}, \mu)$ be a measure space and let $f, g : X \to [0, \infty]$ be measurable. Then

$$\int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu$$
Proof: From superadditivity, we only need to establish
\[ \int_X (f + g) \, d\mu \leq \int_X f \, d\mu + \int_X g \, d\mu. \]

If \( \mu(X) < \infty \) and \( f \) and \( g \) are bounded, given \( \epsilon > 0 \), let \( f^{\epsilon} \) and \( f_\epsilon \) be the simple functions obtained by rounding \( f \) up and down, respectively, to the nearest integer multiple of \( \epsilon \). Then for all \( x \in X \),
\[ f_\epsilon(x) \leq f(x) \leq f^{\epsilon}(x) \]
and
\[ f^{\epsilon}(x) - f_\epsilon(x) \leq \epsilon. \]
We similarly define \( g^{\epsilon} \) and \( g_\epsilon \), so that
\[ f + g \leq f^{\epsilon} + g^{\epsilon} \leq f_\epsilon + g_\epsilon + 2\epsilon, \]
Hence
\[ \int_X (f + g) \, d\mu \leq \text{Simp} \int_X (f_\epsilon + g_\epsilon + 2\epsilon) \, d\mu = \text{Simp} \int_X f_\epsilon \, d\mu + \text{Simp} \int_X g_\epsilon \, d\mu + 2\epsilon \mu(X). \]
The desired result follows on letting \( \epsilon \to 0. \)

If \( \mu(X) < \infty \) but \( f \) and \( g \) are not necessarily bounded, one can use vertical truncations to reduce the problem to the above case and then take the limit.

If \( \mu(X) = \infty \) and one of \( \int_X f \, d\mu \) or \( \int_X g \, d\mu \) is infinite, then by superadditivity so is \( \int_X (f + g) \, d\mu \), from which the desired inequality follows.

Otherwise, if \( \mu(X) = \infty \) but \( \int_X f \, d\mu \) and \( \int_X g \, d\mu \) are both finite, we can conclude from Markov’s inequality that \( S_n = \{ x \in X : f(x) > 1/n \} \cup \{ x \in X : g(x) > 1/n \} \) has finite measure for each \( n \in \mathbb{N} \). Since \( S_n \) are an increasing sequence of measurable sets, we can apply horizontal truncation:
\[ \lim_{n \to \infty} \int_X (f + g) 1_{S_n} \, d\mu = \int_X (f + g) 1_{\bigcup_{n=1}^\infty S_n} \, d\mu = \int_X (f + g) \, d\mu, \]
noting that \( \bigcup_{n=1}^\infty S_n \) is the support of \( f + g \). Our previous results establish that
\[ \int_X (f + g) 1_{S_n} \, d\mu \leq \int_X f 1_{S_n} \, d\mu + \int_X g 1_{S_n} \, d\mu. \]
The desired inequality then follows on taking the limit \( n \to \infty \) and using horizontal truncation.
Problem 1.38: (Linearity in $\mu$) Let $(X, \mathcal{B}, \mu)$ be a measure space and let $f : X \to [0, \infty]$ be measurable. Show that

(i) $\int_X f \, d(c \mu) = c \int_X f \, d\mu$ for every $c \in [0, \infty]$.

(ii) if $\mu_1, \mu_2, \ldots$ are a sequence of measures on $\mathcal{B}$,

$$\int_X f \, d\left(\sum_{k=1}^{\infty} \mu_k\right) = \sum_{k=1}^{\infty} \int_X f \, d\mu_k$$

Problem 1.39 (Sums as integrals): Let $X$ be an arbitrary set, with the discrete $\sigma$-algebra, and $\#$ be the counting measure. Show that every unsigned function $f : X \to [0, \infty]$ is measurable, with

$$\int_X f \, d\# = \sum_{x \in X} f(x).$$

Definition: Let $(X, \mathcal{B}, \mu)$ be a measure space. A measurable function $f : X \to \mathbb{C}$ is said to be absolutely integrable if

$$|f|_{L^1(X,\mathcal{B},\mu)} = \int_X |f| \, d\mu < \infty.$$

Definition: The space of absolutely integrable functions on $(X, \mathcal{B}, \mu)$ is denoted by $L^1(X, \mathcal{B}, \mu)$ or simply $L^1(\mu)$.

Definition: If $f$ is real-valued and absolutely integrable, we define

$$\int_X f \, d\mu \doteq \int_X f_+ \, d\mu - \int_X f_- \, d\mu,$$

where $f_+ \doteq \max(f, 0)$ and $f_- \doteq \max(-f, 0)$.

Definition: If $f$ is complex-valued and absolutely integrable, we define

$$\int_X f \, d\mu \doteq \int_X \text{Re} f \, d\mu + i \int_X \text{Im} f \, d\mu.$$

Theorem 1.11: Let $(X, \mathcal{B}, \mu)$ be a measure space and $f, g \in L^1(X, \mathcal{B}, \mu)$. Then

(i) $L^1(X, \mathcal{B}, \mu)$ is a complex vector space;

(ii) the map $f \mapsto \int_X f \, d\mu$ is a complex-linear map from $L^1(X, \mathcal{B}, \mu)$ to $\mathbb{C}$;

(iii) $|f + g|_{L^1(\mu)} \leq |f|_{L^1(\mu)} + |g|_{L^1(\mu)}$
(iv) $|cf|_{L^1(\mu)} = |c||f|_{L^1(\mu)}$ for every $c \in \mathbb{C}$;

(v) if $f = g \mu$-almost everywhere in $X$, we have $\int_X f \, d\mu = \int_X g \, d\mu$;

(vi) if $(X, B', \mu')$ is a refinement of $(X, B, \mu)$ then $f \in L^1(X, B', \mu')$ and $\int_X f \, d\mu' = \int_X f \, d\mu$;

(vii) $|f|_{L^1(\mu)} = 0$ iff $f$ is zero $\mu$-almost everywhere;

(viii) if $Y \subset X$ is $B$-measurable, then $f_Y \in L^1(Y, B|Y, \mu|Y)$ and $\int_X f \, 1_Y \, d\mu = \int_Y f \, |Y| \, d\mu|Y$.

Q. Under what conditions can we interchange integrals and limits? That is, given a sequence of measurable functions $f_n$ that converges pointwise $\mu$-almost everywhere to a function $f$, under what conditions does

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X \lim_{n \to \infty} f_n \, d\mu?$$

A. There are a number of possible conditions, which we will discuss one at a time.

**Theorem 1.12** (Uniform convergence on finite spaces): Suppose that $(X, B, \mu)$ is a finite measure space ($\mu(X) < \infty$) and $f_n : X \to [0, \infty]$ (or $\mathbb{C}$) is a sequence of measurable (or absolutely integrable) functions that converges uniformly to a limit $f$. Then $\int_X f_n \, d\mu$ converges to $\int_X f \, d\mu$.

**Remark:** If we relax the finite measure or uniformity conditions, it is easy to construct examples in which the interchange of limit processes is invalid:

- In $(\mathbb{R}^d, \mathcal{L}[\mathbb{R}^d], m)$, consider that $f_n \doteq 1_{[n, n+1]}$ converges pointwise to 0 but $\int_{\mathbb{R}} f_n = 1$ does not converge to $\int_{\mathbb{R}} f = 0$. We say that the “mass” of the functions $f_n$ “escapes to horizontal infinity.”

- In $(\mathbb{R}^d, \mathcal{L}[\mathbb{R}^d], m)$, consider that $f_n \doteq \frac{1}{n} 1_{[0, n]}$ converges uniformly to 0 but $\int_{\mathbb{R}} f_n = 1$ does not converge to $\int_{\mathbb{R}} f = 0$. We say that the mass of the functions $f_n$ “escapes to width infinity.”

- In $(\mathbb{R}^d, \mathcal{L}[\mathbb{R}^d], m)$, consider that $f_n \doteq n 1_{[\frac{1}{n}, \frac{2}{n}]}$ converges pointwise to 0 but $\int_{\mathbb{R}} f_n = 1$ does not converge to $\int_{\mathbb{R}} f = 0$. We say that the mass of the functions $f_n$ “escapes to vertical infinity.”
Remark: One way to prevent these three avenues of escape to infinity is to enforce monotonicity; this prevents each function \( f_n \) from “abandoning” the location where the mass of its predecessors was concentrated.

**Theorem 1.13** (Monotone convergence theorem): monotone convergence theorem

Let \((X, \mathcal{B}, \mu)\) be a measure space and \( f_1 \leq f_2 \leq \ldots \) be an increasing sequence of unsigned measurable functions on \( X \). Then

\[
\int_X \lim_{n \to \infty} f_n \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu.
\]

Proof: Let \( f = \lim_{n \to \infty} f_n \), which by Problem 1.36 (vi) is measurable. Let \( g \leq f \) be any simple unsigned function. By applying vertical truncation, we can assume that \( g \) is finite everywhere, so that \( g = \sum_{k=1}^{m} c_k 1_{A_k} \) for some \( c_k \in [0, \infty) \) and disjoint \( \mathcal{B} \)-measurable sets \( A_1, \ldots, A_m \). Thus

\[
\int_X g \, d\mu = \sum_{k=1}^{m} c_k \mu(A_k).
\]

For each \( x \in A_k \) we know

\[
\sup_n f_n(x) = f(x) \geq g(x) = c_k.
\]

Let \( \epsilon \in (0, 1) \). Since the sets

\[
A_{k,n} = \{x \in A_k : f_n(x) > (1 - \epsilon)c_k\}
\]

increase in \( n \) to \( A_k \) and are measurable, we know from upwards monotonicity that

\[
\lim_{n \to \infty} \mu(A_{k,n}) = \mu(A_k).
\]

Moreover, on integrating the inequality

\[
f_n > (1 - \epsilon) \sum_{k=1}^{m} c_k 1_{A_{k,n}},
\]

we find

\[
\int f_n \, d\mu \geq (1 - \epsilon) \sum_{k=1}^{m} c_k \mu(A_{k,n}),
\]

On taking the limit as \( n \to \infty \), we find

\[
\lim_{n \to \infty} \int f_n \, d\mu \geq (1 - \epsilon) \sum_{k=1}^{m} c_k \mu(A_k) = (1 - \epsilon) \int g \, d\mu.
\]
Since $\epsilon \in (0, 1)$ is arbitrary, it follows that
$$
\lim_{n \to \infty} \int_X f_n \, d\mu \geq \int_X g \, d\mu.
$$
On taking the supremum over all simple functions $g \leq f$, we find
$$
\lim_{n \to \infty} \int_X f_n \, d\mu \geq \int_X f \, d\mu.
$$
Since the reverse inequality holds by monotonicity:
$$
\lim_{n \to \infty} \int_X f_n \, d\mu \leq \int_X f \, d\mu,
$$
we arrive at the desired result.

**Corollary 1.13.1** (Tonelli’s theorem for sums and integrals): Let $(X, \mathcal{B}, \mu)$ be a measure space and $f_1, f_2, \ldots$ be a sequence of unsigned measurable functions on $X$. Then
$$
\int_X \sum_{k=1}^{\infty} f_k \, d\mu = \sum_{k=1}^{\infty} \int_X f_k \, d\mu.
$$

**Proof:** Apply Theorem 1.13 to the partial sums $\sum_{k=1}^{n} f_k$.

**Corollary 1.13.2** (Borel–Cantelli lemma): Let $(X, \mathcal{B}, \mu)$ be a measure space and $S_1, S_2, \ldots$ be a sequence of $\mathcal{B}$-measurable sets such that $\sum_{k=1}^{\infty} \mu(S_k) < \infty$. Then for $\mu$-almost every $x \in X$ the set $\{k \in \mathbb{N} : x \in S_k\}$ is finite.

**Proof:** Apply Tonelli’s theorem to $1_{S_k}$.

**Remark:** When one does not have monotonicity, Fatou’s lemma at least provides an inequality:

**Corollary 1.13.3** (Fatou’s lemma): Let $(X, \mathcal{B}, \mu)$ be a measure space and $f_1, f_2, \ldots$ be a sequence of unsigned measurable functions on $X$. Then
$$
\int_X \liminf_{n \to \infty} f_n \, d\mu \leq \liminf_{n \to \infty} \int_X f_n \, d\mu.
$$

**Proof:** By definition,
$$
\liminf_{n \to \infty} f_n = \lim_{n \to \infty} F_n,
$$
where $F_n = \inf_{k \geq n} f_k$. Monotonicity implies that $\int_X F_n \, d\mu \leq \int_X f_k \, d\mu$ for all $k \geq n$; hence,
$$
\int_X F_n \, d\mu \leq \inf_{k \geq n} \int_X f_k \, d\mu.
$$
Since $\{F_n\}_{n=1}^{\infty}$ is an increasing sequence of measurable functions, we know by Theorem 1.13 that
$$
\int_X \lim_{n \to \infty} F_n \, d\mu = \lim_{n \to \infty} \int_X F_n \, d\mu \leq \liminf_{n \to \infty} \int_X f_k \, d\mu = \liminf_{n \to \infty} \int_X f_n \, d\mu.
$$
Remark: Fatou’s lemma tells us that while the “mass” \( \int_X f_n \, d\mu \) can be destroyed in taking pointwise limits, as we saw in the three escapes to infinity, it cannot be created.

Definition: Let \((X, \mathcal{B}, \mu)\) be a measure space and \((Y, \mathcal{C})\) be a measurable space. A measurable morphism is a function \( \phi : X \to Y \) such that \( \phi^{-1}(S) \) is \( \mathcal{B} \)-measurable for every \( \mathcal{C} \)-measurable set \( S \).

Remark: Let \((X, \mathcal{B}, \mu)\) be a measure space, \((Y, \mathcal{C})\) be a measurable space, and \( \phi \) be a measurable morphism from \( X \) to \( Y \). The pushforward \( \phi_* \mu(S) = \mu(\phi^{-1}(S)) \) is a measure on \( \mathcal{C} \), so that \((Y, \mathcal{C}, \phi_* \mu)\) is a measure space.

- If \( T : \mathbb{R}^d \to \mathbb{R}^d \) is an invertible linear transformation, then \( T_* m = \frac{1}{|\det T|} m \).

Corollary 1.13.4 (Change of variables): Let \((X, \mathcal{B}, \mu)\) be a measure space, and \( \phi \) be a measurable morphism from \( X \) to \( Y \). If \( f : Y \to [0, \infty] \) is measurable, then
  \[
  \int_Y f \, d\phi_* \mu = \int_X f \circ \phi \, d\mu.
  \]

Remark: Another important way to avoid loss of mass in taking pointwise limits is to dominate all of the functions by an absolutely integrable one:

Theorem 1.14 (Dominated convergence theorem): Let \((X, \mathcal{B}, \mu)\) be a measure space and \( f_1, f_2, \ldots \) be a sequence of complex-valued measurable functions on \( X \) that converge pointwise \( \mu \)-almost everywhere on \( X \). If there exists an unsigned absolutely integrable function \( G : X \to [0, \infty] \) such that \( |f_n| \leq G \) \( \mu \)-almost everywhere, then
  \[
  \int_X \lim_{n \to \infty} f_n \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu.
  \]

Proof: Without loss of generality, by considering real and imaginary parts, we may assume that the functions \( f_n \) are real and modify them on null sets so that \( \mu \)-almost “everywhere” becomes “everywhere”. Then \(-G \leq f_n \leq G\). Let \( f = \lim_{n \to \infty} f_n \).

On applying Fatou’s lemma to \( f_n + G \) and \( G - f_n \), we find
  \[
  \int_X (f + G) \, d\mu \leq \liminf_{n \to \infty} \int_X (f_n + G) \, d\mu
  \]
  and
  \[
  \int_X (G - f) \, d\mu \leq \liminf_{n \to \infty} \int_X (G - f_n) \, d\mu.
  \]
Since \( G \) is absolutely integrable, we may subtract the finite quantity \( \int_X G \, d\mu \) from both sides of these equations to obtain
  \[
  \limsup_{n \to \infty} \int_X f_n \, d\mu \leq \int_X f \, d\mu \leq \liminf_{n \to \infty} \int_X f_n \, d\mu.
  \]
The desired result then follows from the fact that \( \liminf_{n \to \infty} \int_X f_n \, d\mu \leq \limsup_{n \to \infty} \int_X f_n \, d\mu \).
1.G Modes of convergence

**Definition:** A sequence of functions $f_n : X \to \mathbb{C}$ converges pointwise almost everywhere to $f : X \to \mathbb{C}$ if for every $\epsilon > 0$ and almost every $x \in X$ there exists $N = N(\epsilon, x)$ such that $|f_n(x) - f(x)| < \epsilon$ whenever $n \geq N$.

**Definition:** A sequence of functions $f_n : X \to \mathbb{C}$ converges uniformly almost everywhere (or in $L^\infty$ norm) to $f : X \to \mathbb{C}$ if for every $\epsilon > 0$ there exists $N = N(\epsilon)$ such that $|f_n(x) - f(x)| < \epsilon$ for almost every $x \in X$ whenever $n \geq N$.

**Definition:** A sequence of functions $f_n : X \to \mathbb{C}$ converges almost uniformly to $f : X \to \mathbb{C}$ if for every $\epsilon > 0$ there exists an exceptional set $E_\epsilon$ of measure less than $\epsilon$ such that $f_n$ converges uniformly to $f$ on $X \setminus E_\epsilon$.

**Definition:** A sequence of functions $f_n : X \to \mathbb{C}$ converges in the $L^1$ norm if $|f_n - f|_{L^1}$ converges to 0 as $n \to \infty$.

**Definition:** A sequence of functions $f_n : X \to \mathbb{C}$ converges in measure $\mu$ if for every $\epsilon > 0$, the measures $\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\})$ converge to 0 as $n \to \infty$.

- In $(\mathbb{R}^d, L[\mathbb{R}^d], m)$, we see that $f_n \doteq 1_{[n,n+1]}$ converges pointwise to 0 but not uniformly, in the $L^\infty$ or $L^1$ norms, almost uniformly, or in measure.

- In $(\mathbb{R}^d, L[\mathbb{R}^d], m)$, we see that $f_n \doteq \frac{1}{n} 1_{[0,n]}$ converges uniformly to 0 (and hence pointwise, in the $L^\infty$ norm, almost uniformly, and in measure) but not in the $L^1$ norm.

- In $(\mathbb{R}^d, L[\mathbb{R}^d], m)$, we see that $f_n \doteq n 1_{[\frac{1}{n}, \frac{2}{n}]}$ converges to 0 pointwise almost uniformly (and hence in measure), but not uniformly or in the $L^\infty$ or $L^1$ norms.

- In $(\mathbb{R}^d, L[\mathbb{R}^d], m)$, the typewriter sequence
  
  $$f_n = 1_{[\frac{n-2k_2}{2^k}, \frac{n-2k_1+1}{2^k}]} \quad \text{for } k \geq 0 \text{ and } n \in [2^k, 2^{k+1} - 1]$$

  converges to zero in measure and in the $L^1$ norm, but not pointwise almost everywhere, almost uniformly, or in the $L^\infty$ norm.

**Remark:** The $L^\infty$ norm $|f|_{L^\infty}$ of a measurable function is the infimum of all $M \in [0, \infty]$ such that $|f| \leq M$ for almost all $x$. 

Remark: The five modes of convergence are all compatible in the sense that, outside of a set of measure zero, they never disagree about which function a sequence of functions converges to.

Remark: If a sequence of absolutely integrable functions $f_n$ converges to $f$ in the $L^1$ norm, the triangle inequality implies that
\[
\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.
\]

Definition: We say that a sequence $f_n : X \to \mathbb{C}$ is dominated if there exists an absolutely integrable function $g : X \to [0, \infty]$ such that $|f_n(x)| \leq g(x)$ for all $n$ and almost every $x \in X$.

Problem 1.40: If a dominated sequence $f_n : X \to \mathbb{C}$ converges pointwise almost everywhere, use the dominated convergence theorem to show that it converges in the $L^1$ norm.

Definition: A sequence $f_n : X \to \mathbb{C}$ of absolutely integrable functions is said to be uniformly integrable if the following three statements all hold:

(i) (Uniform bound on $L^1$ norm) One has $\sup_n |f_n|_{L^1} < \infty$.

(ii) (No escape to vertical infinity) One has $\sup_n \int_{|f_n| \geq M} |f_n| \, d\mu \to 0$ as $M \to \infty$.

(iii) (No escape to width infinity) One has $\sup_n \int_{|f_n| \leq \delta} |f_n| \, d\mu \to 0$ as $\delta \to 0$.

Remark: Given an absolutely integrable function $f$, we can apply the monotone convergence theorem to $|f|1_{B_{1/n}[0]}$ for $n = 1, 2, \ldots$ and $|f|1_{B_m[0]}$ to conclude that the constant sequence of functions $f$ is uniformly integrable.

Theorem 1.15 (Uniformly integrable convergence in measure): Let $f_n : X \to \mathbb{C}$ be a uniformly integrable sequence of functions, and let $f : X \to \mathbb{C}$ be another function. Then $f_n$ converges in $L^1$ norm iff $f_n$ converges to $f$ in measure.

Proof: See Tao, Theorem 1.5.13.

1.H Differentiation theorems
**Definition:** Let \([a, b]\) be a compact interval of positive length. A function \(F : [a, b] \to \mathbb{R}\) is almost-everywhere differentiable if the limit
\[
F'(x) = \lim_{y \to x, y \in [a, b] \{x\}} \frac{F(y) - F(x)}{y - x}
\]
exists for almost all \(x \in [a, b]\).

**Problem 1.41:** Provide an example that illustrates that an almost-everywhere differentiable function need not be continuous everywhere.

**Problem 1.42:** If \(F : [a, b] \to \mathbb{R}\) is almost-everywhere differentiable, show that \(F\) is continuous almost everywhere.

**Problem 1.43:** If \(F : [a, b] \to \mathbb{R}\) is almost-everywhere differentiable, show that (the almost-everywhere defined derivative) \(F'\) is measurable.

*Hint:* \(F'(x) = \lim_{n \to \infty} n(F(x + 1/n) - F(x))\).

**Definition:** If \(F\) is differentiable and its derivative \(F'\) is continuous, we say that \(F\) is continuously differentiable.

**Problem 1.44:** Let \(f : \mathbb{R} \to \mathbb{C}\) be an absolutely integrable function. Show that the indefinite integral \(F(x) \doteq \int_{-\infty}^{x} f(t) \, dt\) is a continuous function on \(\mathbb{R}\).

**Remark:** In order to extend the fundamental theorem of calculus to the Lebesgue integral, we first need to establish some preliminary results.

**Lemma 1.17** (Rising sun lemma): Let \(F\) be a real-valued continuous function on \([a, b]\) and \(S = \{x \in [a, b] : F(x) < F(y) \text{ for some } y \in (x, b]\}\). Define \(U = S \cap (a, b)\). Then \(U\) is open and may be written as a countable union of disjoint intervals \(U = \bigcup_k (a_k, b_k)\) such that \(F(a_k) = F(b_k)\), unless \(a_k = a \in S\) for some \(k\), in which case \(F(a) < F(b_k)\) for that one \(k\). Furthermore, if \(x \in (a_k, b_k)\), then \(F(x) < F(b_k)\).

**Remark:** If we imagine the graph of the function \(F\) as a hilly landscape, with the sun shining horizontally from the right (rising from the east), the set \(U\) consist of those points that are in shadow. Proof:
CHAPTER 1. MEASURE THEORY

Claim: If \([c, d) \subset S\), with \(d \notin S\), then \(F(c) < F(d)\). Otherwise, suppose \(F(c) \geq F(d)\). Then \(F\) achieves its maximum on \([c, d]\) at some point \(x < d\). Since \(x \in S\), we know that \(F(x) < F(y)\) for some \(y \in (x, b]\). But \(F(x) < F(y)\) implies that \(y \notin [c, d]\). Hence \(y \in (d, b]\) and \(F(d) \leq F(x) < F(y)\), contradicting \(d \notin S\).

Since \(F\) is continuous, \(U\) is open and can be expressed as a countable union of disjoint intervals \((a_k, b_k)\).

Since \(b_k \notin S\), the claim establishes that \(x \in (a_k, b_k) \Rightarrow F(x) < F(b_k)\). Since \(F\) is continuous, \(F(a_k) \leq F(b_k)\).

If \(a_k = a \in S\), the claim tells us that \(F(a) < F(b_k)\).

Otherwise if \(a \notin S\), then \(a_k \notin S\), so \(F(a_k) \geq F(b_k)\). Thus, \(F(a_k) = F(b_k)\).

**Lemma 1.18** (One-sided Hardy-Littlewood maximal inequality): Let \(f : \mathbb{R} \to \mathbb{C}\) be an absolutely integrable function and let \(\lambda > 0\). Then

\[
m\left( \left\{ x \in \mathbb{R} : \sup_{h > 0} \frac{1}{h} \int_{[x, x+h]} |f(t)| \, dt \geq \lambda \right\} \right) \leq \frac{1}{\lambda} \int_{\mathbb{R}} |f(t)| \, dt.
\]

Proof: Let \([a, b]\) be any compact interval and define

\[
S_\lambda = \left\{ x \in [a, b] : \sup_{h > 0} \frac{1}{h} \int_{[x, x+h]} |f(t)| \, dt > \lambda \right\}.
\]

We first establish for any \(\epsilon \in (0, \lambda)\) that

\[
m(S_{\lambda-\epsilon}) \leq \frac{1}{\lambda - \epsilon} \int_{\mathbb{R}} |f(t)| \, dt.
\]

For \(x \in [a, b]\), consider the continuous function

\[
F(x) = \int_{[a, x]} |f(t)| \, dt - x(\lambda - \epsilon)
\]

and note that the inequality \(\frac{1}{h} \int_{[x, x+h]} |f(t)| \, dt > \lambda - \epsilon\) reduces to \(F(x + h) > F(x)\).

On applying the rising sun lemma to \(F\), we see that there exists a countable sequence of disjoint intervals \(\{(a_k, b_k)\}_{k=1}^\infty\) such that

\[
S_{\lambda-\epsilon} \subset \bigcup_{k=1}^\infty (a_k, b_k) \cup \{a\} \cup \{b\}.
\]

By countable additivity and monotonicity, we then find that

\[
m(S_{\lambda-\epsilon}) \leq \sum_{k=1}^\infty (b_k - a_k).
\]
The rising sun lemma also tells us that

\[ 0 \leq F(b_k) - F(a_k) = \int_{[a_k,b_k]} |f(t)| \, dt - (b_k - a_k)(\lambda - \epsilon), \]

so

\[ m(S_{\lambda-\epsilon}) \leq \sum_{k=1}^{\infty} (b_k - a_k) \leq \frac{1}{\lambda-\epsilon} \sum_{k=1}^{\infty} \int_{[a_k,b_k]} |f(t)| \, dt \leq \frac{1}{\lambda-\epsilon} \int_{[a,b]} |f(t)| \, dt, \]

where we have exploited additivity and monotonicity.

Finally, since

\[ \left\{ x \in [a,b] : \sup_{h>0} \frac{1}{h} \int_{[x,x+h]} |f(t)| \, dt \geq \lambda \right\} \subset S_{\lambda-\epsilon}, \]

the desired result then follows on letting \( \epsilon \to 0 \) and then applying upward monotonicity.

**Theorem 1.16** (Lebesgue differentiation theorem on \( \mathbb{R} \)): Let \( f : \mathbb{R} \to \mathbb{C} \) be an absolutely integrable function. Then

\[ \lim_{h \to 0^+} \frac{1}{h} \int_{[x,x+h]} f(t) \, dt = f(x) \]

and

\[ \lim_{h \to 0^+} \frac{1}{h} \int_{[x-h,x]} f(t) \, dt = f(x) \]

for almost every \( x \in \mathbb{R} \).

**Proof:** Let \( \epsilon > 0 \). By Littlewood’s second principle, there exists a continuous, compactly supported function \( g : \mathbb{R} \to \mathbb{C} \) such that

\[ \int_{\mathbb{R}} |f(t) - g(t)| \, dt \leq \epsilon. \]

Since \( g \) is continuous, we know from the fundamental theorem of calculus that

\[ \lim_{h \to 0^+} \frac{1}{h} \int_{[x,x+h]} g(t) \, dt = g(x). \]

Let \( n \in \mathbb{N} \). Then there exists an \( h > 0 \) such that

\[ \left| \frac{1}{h} \int_{[x,x+h]} g(t) \, dt - g(x) \right| < \frac{1}{n}. \]
From the Hardy-Littlewood maximal inequality we know that
\[
m\left(\left\{ x \in \mathbb{R} : \sup_{h>0} \frac{1}{h} \int_{[x,x+h]} |f(t) - g(t)| \, dt \geq \frac{1}{n} \right\}\right) \leq n \epsilon.
\]
Likewise, Markov’s inequality implies that
\[
m\left(\left\{ x \in \mathbb{R} : |f(x) - g(x)| \geq \frac{1}{n} \right\}\right) \leq n \epsilon.
\]
By subadditivity, the measure of the union of the sets on the left-hand sides of the above two inequalities is at most \(2n \epsilon\). For \(x\) outside of this union and all \(h > 0\),
\[
\frac{1}{h} \int_{[x,x+h]} |f(t) - g(t)| \, dt < \frac{1}{n} \quad \text{and} \quad |f(x) - g(x)| < \frac{1}{n}.
\]
For such \(x\), we then deduce from the triangle inequality that
\[
\limsup_{h \to 0^+} \left| \frac{1}{h} \int_{[x,x+h]} f(t) \, dt - f(x) \right| < \frac{3}{n}.
\]
On taking the limits as \(\epsilon \to 0\) and then \(n \to \infty\), we see for almost all real \(x\) that
\[
\limsup_{h \to 0^+} \left| \frac{1}{h} \int_{[x,x+h]} f(t) \, dt - f(x) \right| = 0.
\]
Since the corresponding limit inferior is non-negative, the first statement in the theorem then follows. The second statement follows on applying the first statement to the reflected function \(x \mapsto f(-x)\).

**Corollary 1.16.1:** Let \(f : \mathbb{R} \to \mathbb{C}\) be an absolutely integrable function, and let \(F : \mathbb{R} \to \mathbb{C}\) be the indefinite integral \(F(x) = \int_{[\mathbb{R},x]} f(t) \, dt\). Then \(F\) is continuous and almost everywhere differentiable, with \(F'(x) = f(x)\) for almost every \(x \in \mathbb{R}\).

**Remark:** The Lebesgue differentiation theorem has an analogue in higher dimensions.

**Definition:** Let \(f : \mathbb{R}^d \to \mathbb{C}\) be an absolutely integrable function. A point \(x\) where
\[
\lim_{r \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| \, dy = 0.
\]
is known as a **Lebesgue point**.

**Theorem 1.17** (Lebesgue differentiation theorem on \(\mathbb{R}^d\)): Let \(f : \mathbb{R}^d \to \mathbb{C}\) be an absolutely integrable function. Then almost every \(x \in \mathbb{R}^d\) is a Lebesgue point for \(f\).
Corollary 1.17.1: Let \( f : \mathbb{R}^d \to \mathbb{C} \) be an absolutely integrable function. Then

\[
\lim_{r \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y) \, dy = f(x)
\]

for almost every \( x \in \mathbb{R}^d \).

Proof: Apply the triangle inequality to Theorem 1.17.

Theorem 1.18 (Monotone differentiation theorem): Every monotone function \( f : \mathbb{R} \to \mathbb{R} \) is differentiable almost everywhere.

Proof: See Tao, Theorem 1.6.25.

Definition: The total variation of a function \( F : \mathbb{R} \to \mathbb{R} \) on an (finite or infinite) interval \( I \) is

\[
|F|_{TV(I)} = \sup_{x_0 < \ldots < x_n \in I} \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})|.
\]

If \( |F|_{TV(I)} \) is finite, we say that \( F \) has bounded variation on \( I \). If \( F \) has bounded variation on \( \mathbb{R} \), we say that \( F \) has bounded variation.

Problem 1.45: If \( F : \mathbb{R} \to \mathbb{R} \) is a monotone function, show that \( |F|_{TV([a,b])} = |F(b) - F(a)| \) for any interval \([a,b]\). Conclude that \( F \) has bounded variation on \( \mathbb{R} \) iff it is bounded.

Problem 1.46: For any functions \( F, G : \mathbb{R} \to \mathbb{R} \) and \( c \in \mathbb{R} \), show that \( |F + G|_{TV(\mathbb{R})} \leq |F|_{TV(\mathbb{R})} + |G|_{TV(\mathbb{R})} \) and \( |cF|_{TV(\mathbb{R})} = |c||F|_{TV(\mathbb{R})} \).

Problem 1.47: If \( F : \mathbb{R} \to \mathbb{R} \) is a function, show that \( |F|_{TV([a,b])} + |F|_{TV([b,c])} = |F|_{TV([a,c])} \) whenever \( a \leq b \leq c \).

Problem 1.48:

(i) Show that every function \( F : \mathbb{R} \to \mathbb{R} \) of bounded variation is bounded and that \( \lim_{x \to \infty} F(x) \) and \( \lim_{x \to -\infty} F(x) \) exist.

(ii) Provide a counterexample of a bounded, continuous function \( F \) with bounded support that does not have bounded variation.

Theorem 1.19: A function \( F : \mathbb{R} \to \mathbb{R} \) has bounded variation iff it is the difference of two bounded monotone functions.
Proof: “⇐” This follows from Prob 1.45 and 1.46.

“⇒” Define the positive variation of \( F \) to be the bounded increasing function

\[
F^+(x) = \sup_{x_0 < \ldots < x_n \leq x} \sum_{i=1}^{n} \max(F(x_i) - F(x_{i-1}), 0).
\]

We claim that \( F^+ - F \) is monotone increasing: that is,

\[
F^+(b) \geq F^+(a) + F(b) - F(a)
\]

whenever \( b \geq a \). If \( F(b) - F(a) < 0 \), this follows from the monotonicity of \( F^+ \). Otherwise, if \( F(b) - F(a) \geq 0 \), one can add \( a \) and \( b \) to any sequence \( x_0 < \ldots < x_n \leq a \), which will increase

\[
\sup_{x_0 < \ldots < x_n} \sum_{i=1}^{n} \max(F(x_i) - F(x_{i-1}), 0)
\]

by at least \( F(b) - F(a) \), thereby establishing the claim. The result then follows from the observation that \( F = F^+ - (F^+ - F) \).

**Corollary 1.19.1** (BV differentiation theorem): A function of bounded variation is differentiable almost everywhere.

**Definition**: A function \( f : \mathbb{R} \to \mathbb{R} \) is locally of bounded variation if it has bounded variation on every compact interval of \( \mathbb{R} \).

**Remark**: A function that is locally of bounded variation is differentiable almost everywhere.

**Definition**: A function \( f : \mathbb{R} \to \mathbb{R} \) is said to be Lipschitz continuous if there exists a positive constant \( C \) such that \( |f(x) - f(y)| \leq C|x - y| \) for all \( x, y \in \mathbb{R} \). The smallest \( C \) with this property is known as the Lipschitz constant of \( f \).

**Theorem 1.20** (1D Lipschitz differentiation theorem): Every Lipschitz continuous function is locally of bounded variation, and hence differentiable almost everywhere. Furthermore, its derivative, when it exists, is bounded by its Lipschitz constant.

**Problem 1.49**: Show that every convex function \( f : \mathbb{R} \to \mathbb{R} \) is continuous and almost everywhere differentiable, with derivative almost everywhere equal to an increasing function.

**Theorem 1.21** (Upper bound for fundamental theorem): Let \( F : [a, b] \to \mathbb{R} \) be increasing, so that the unsigned function \( F' : [a, b] \to [0, \infty] \) exists almost everywhere and is measurable. Then

\[
\int_{[a,b]} F' \leq F(b) - F(a).
\]
Proof: Extend $F$ to $\mathbb{R}$ by defining $F(x) = F(a)$ for $x < a$ and $F(x) = F(b)$ for $x > b$, so that $F$ is a bounded monotone function on $\mathbb{R}$ such that $F'$ vanishes outside of $[a, b]$. As $F$ is almost everywhere differentiable, the sequence of functions

$$f_n(x) = \frac{F(x + 1/n) - F(x)}{1/n}$$

converges pointwise almost everywhere to $F'$. Apply Fatou’s lemma to conclude that

$$\int_{[a,b]} F'(x) \, dx \leq \liminf_{n \to \infty} \int_{[a,b]} f_n \, dx = \liminf_{n \to \infty} n \int_{[a,b]} \left[F\left(x + \frac{1}{n}\right) - F(x)\right] \, dx$$

$$= \liminf_{n \to \infty} \left[ \int_{[a+1/n,b+1/n]} F(x) \, dx - \int_{[a,b]} F(x) \, dx \right]$$

$$= \liminf_{n \to \infty} \left[ \int_{[b,b+1/n]} F(x) \, dx - \int_{[a,a+1/n]} F(x) \, dx \right]$$

$$\leq \liminf_{n \to \infty} \left[ F(b) \int_{[b,b+1/n]} \, dx - F(a) \int_{[a,a+1/n]} \, dx \right]$$

$$= F(b) - F(a).$$

Remark: In Theorem 1.21 we note that $F'$ is absolutely integrable. This implies that every function of bounded variation has an almost-everywhere defined derivative that is absolutely integrable.

Problem 1.50: Prove that the product of two Lipschitz continuous functions is itself Lipschitz continuous.

Problem 1.51 (Integration by parts): Let $F, G : [a, b] \to \mathbb{R}$ be Lipschitz continuous functions. Show that

$$\int_{[a,b]} F'G = F(b)G(b) - F(a)G(a) - \int_{[a,b]} FG'$$

Definition: A function $F : \mathbb{R} \to \mathbb{R}$ is said to be absolutely continuous if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\sum_{k=1}^{n} |F(b_k) - F(a_k)| < \epsilon$ for every finite collection of disjoint intervals $(a_1, b_1) \ldots (a_n, b_n)$ of total length $\sum_{k=1}^{n} (b_k - a_k) < \delta$.

• The function $x \to \sqrt{x}$ is absolutely continuous, but not Lipschitz continuous, on the interval $[0, 1]$. 

The Cantor function is continuous, monotone, and uniformly continuous, but not absolutely continuous, on \([0,1]\).

**Problem 1.52:** Show that

(i) every absolutely continuous function is uniformly continuous (and therefore continuous);

(ii) every absolutely continuous function is of bounded variation (and hence differentiable almost everywhere) on every compact interval \([a,b]\) (hint: first show this is true for every sufficiently small interval);

(iii) every Lipschitz continuous function is absolutely continuous;

(iv) if \(f : \mathbb{R} \to \mathbb{R}\) is absolutely integrable, the indefinite integral \(F(x) \doteq \int_{[-\infty,x]} f(y) \, dy\) is absolutely continuous and that \(F\) is differentiable almost everywhere, with \(F'(x) = f(x)\) for almost every \(x\);

(v) the sum and product of two absolutely continuous functions on an interval \([a,b]\) is absolutely continuous (what happens if we replace \([a,b]\) by \(\mathbb{R}\))?

**Problem 1.53** (1D Besicovitch covering lemma): Let \(I_1, \ldots, I_n\) be a finite family of open intervals in \(\mathbb{R}\) (not necessarily disjoint). Show that there exist a subfamily \(I'_1, \ldots, I'_m\) of intervals such that \(\bigcup_{i=1}^n I_i = \bigcup_{j=1}^m I'_j\) and every point of \(\mathbb{R}\) is contained in at most two of the intervals \(I'_j\). Hint: First refine the family of intervals so that no interval \(I_i\) is contained in the union of the the other intervals. Then show that it is no longer possible for a point to be contained in three of the intervals.

**Problem 1.54** (Cousin’s theorem): Given any (so-called gauge) function \(\delta : [a,b] \to (0,\infty)\) on a compact interval \([a,b]\) of positive length, show that there exists a partition \(a = t_0 < \ldots < t_n = b\) of \([a,b]\), where \(n \in \mathbb{N}\), and real numbers \(t^*_k \in [t_{k-1}, t_k]\) for each \(k = 1, \ldots, n\) such that \(t_k - t_{k-1} \leq \delta(t^*_k)\). Hint: use the Heine–Borel theorem and the Besicovitch covering lemma.

**Theorem 1.22** (Fundamental theorem for absolutely continuous functions): Let \(F : [a,b] \to \mathbb{R}\) be absolutely continuous. Then

\[
\int_{[a,b]} F' = F(b) - F(a).
\]

Proof: Let \(\epsilon > 0\). Since \(F\) is absolutely continuous, there exists \(N_1 \in \mathbb{N}\) such that \(\sum_{k=1}^n |F(b_k) - F(a_k)| < \epsilon\) for every finite collection of disjoint intervals \((a_1, b_1) \ldots (a_n, b_n)\) of total length \(\sum_{k=1}^n (b_k - a_k) < 1/N_1\).
Thus there has an almost-everywhere defined derivative and note that monotonicity implies $m$ measure

By outer regularity, for each $n \in \mathbb{N}$, there exists an open set $U_n$ containing $E$ with measure $m(U_n) < 1/n$. Consider the decreasing sequence of open sets $V_n = \cap_{k=1}^n U_k$ and note that monotonicity implies $m(V_n) < 1/n$.

We also know that the absolutely continuous function $F$ is of bounded variation and therefore has an almost-everywhere defined derivative $F'$ that is absolutely integrable. Thus

$$\int_{[a,b] \setminus V_n} |F'| + \int_{V_n} |F'| = \int_{[a,b]} |F'| < \infty.$$  

On applying the monotone convergence theorem, we then see that

$$\lim_{n \to \infty} \int_{V_n} |F'| = 0.$$  

In particular, there exists $N_2 \in \mathbb{N}$ such that $\int_{V_n} |F'| < \epsilon$ whenever $n \geq N_2$.

Let $N = \max(N_1, N_2)$ and define a gauge function $\delta : [a, b] \to (0, \infty)$:

- if $x \in E$, we choose $\delta(x)$ small enough so that $(x - \delta(x), x + \delta(x))$ is contained in the open set $V_N$;

- if $x \notin E$, then $F$ is differentiable at $x$ and $x$ is a Lebesgue point of $F'$, so we can choose $\delta(x)$ small enough such that $|y - x| < \delta(x)$ implies

$$|F(y) - F(x) - (y - x)F'(x)| \leq \epsilon|y - x|$$

and

$$\left| \frac{1}{|I|} \int_I F'(y) \, dy - F'(x) \right| < \epsilon$$

whenever $I$ is an interval of length at most $\delta(x)$ containing $x$.

By Cousin’s theorem, we can find a partition $a = t_0 < \ldots < t_n = b$ of $[a, b]$, where $n \in \mathbb{N}$, and real numbers $t_k^* \in [t_{k-1}, t_k]$ for each $k = 1, \ldots, n$ such that $t_k - t_{k-1} \leq \delta(t_k^*)$.

For those $k$ such that $t_k^* \in E$, we note from the choice of $\delta(x)$ that the disjoint intervals $(t_{k-1}, t_k)$ are all contained in $V_N$, with $\sum_{k:t_k^* \in E}(t_k - t_{k-1}) \leq m(V_N) < 1/N \leq 1/N_1$. Then

$$\sum_{k:t_k^* \in E} |F(t_k) - F(t_{k-1})| < \epsilon.$$  

This implies that

$$\sum_{k:t_k^* \in E} F(t_k) - F(t_{k-1}) = \mathcal{O}(\epsilon).$$

For those $k$ such that $t_k^* \notin E$, we have

$$F(t_k) - F(t_k^*) = (t_k - t_k^*)F'(t_k^*) + \mathcal{O}(\epsilon|t_k - t_k^*|)$$
and
\[ F(t_k^*) - F(t_{k-1}) = (t_k^* - t_{k-1})F'(t_k^*) + \mathcal{O}(\epsilon|t_k^* - t_{k-1}|), \]
so that
\[ F(t_k) - F(t_{k-1}) = (t_k - t_{k-1})F'(t_k^*) + \mathcal{O}(\epsilon|t_k - t_{k-1}|), \]
We also know that
\[ \int_{[t_{k-1}, t_k]} F' = (t_k - t_{k-1})F'(t_k^*) + \mathcal{O}(\epsilon|t_k - t_{k-1}|). \]
On combining these two results, we find
\[ F(t_k) - F(t_{k-1}) = \int_{[t_{k-1}, t_k]} F' + \mathcal{O}(\epsilon|t_k - t_{k-1}|). \]
Let \( S \subset [a, b] \) be the union of all \([t_{k-1}, t_k]\) with \( t_k^* \notin E \). Then
\[ \sum_{k: t_k^* \notin E} [F(t_k) - F(t_{k-1})] = \int_S F' + \mathcal{O}(\epsilon|b - a|). \]
Since \( S \) contains \([a, b] \setminus V_N\) and \( \int_{V_N} |F'| < \epsilon \), we see that
\[ \int_S F' = \int_{[a,b]} F' + \mathcal{O}(\epsilon). \]
These observations together then give us an expression for the telescoping sum
\[ F(b) - F(a) = \sum_{k=1}^n [F(t_k) - F(t_{k-1})] = \int_{[a,b]} F' + \mathcal{O}(\epsilon) + \mathcal{O}(\epsilon|b - a|), \]
from which the desired result follows.

1.I Outer measures, pre-measures, and product measures

**Definition:** Given a set \( X \), an outer measure is a map \( \mu^*: 2^X \mapsto [0, \infty] \) such that

(i) \( \mu^*(\emptyset) = 0; \) \hspace{2cm} **nullity**

(ii) \( S \subset T \subset X \Rightarrow \mu^*(S) \leq \mu^*(T); \) \hspace{2cm} **monotonicity**

(iii) \( \mu^*\left(\bigcup_{k=1}^{\infty} S_k\right) \leq \sum_{k=1}^{\infty} \mu^*(S_k) \), where \( S_k \subset X. \) \hspace{2cm} **countable subadditivity**
The Lebesgue outer measure is an outer measure.

The Jordan outer measure is not an outer measure (and for this reason should more properly be called Jordan outer content).

Remark: Although outer measures, requiring only countable subadditivity (rather than countable additivity), are weaker than measures, they provide a measure for all subsets of \( X \) (rather than for just a \( \sigma \)-algebra of measurable sets).

Remark: The concept of an open set, which was used to define the Lebesgue measure, is not available for an arbitrary set \( X \).

Definition: Let \( \mu^* \) be an outer measure on a set \( X \). A set \( S \subset X \) is said to be Carathéodory measurable if

\[
\mu^*(A) = \mu^*(A \cap S) + \mu^*(A \cap S^c);
\]

for every set \( A \subset X \).

Definition: A set \( S \) is a null set for an outer measure \( \mu^* \) if \( \mu^*(S) = 0 \).

Problem 1.55: Suppose \( S \) is a null set for an outer measure \( \mu^* \). Show that \( S \) is Carathéodory measurable with respect to \( \mu^* \).

Since \( A \subset (A \setminus S) \cup S \), we see that \( \mu^*(A) \leq \mu^*(A \setminus S) + \mu^*(S) = \mu^*(A \setminus S) \). But \( A \setminus S \subset A \), so \( \mu^*(A \setminus S) \leq \mu^*(A) \). Hence \( \mu^*(A) = \mu^*(A \setminus S) = \mu^*(A \cap S) + \mu^*(A \cap S^c) \), on applying monotonicity to \( A \cap S \subset S \).

Problem 1.56: Let \( \mathcal{B} \) be a Boolean algebra on a set \( X \). Show that \( \mathcal{B} \) is a \( \sigma \)-algebra iff it is closed under countable disjoint unions.

This follows from the fact for each set \( B \), every set \( A \) can be expressed as the union of the disjoint sets \( A \cap B \) and \( A \cap B^c \).

Theorem 1.23 (Carathéodory lemma): Let \( \mu^* : 2^X \rightarrow [0, \infty] \) be an outer measure on a set \( X \), let \( \mathcal{B} \) be the collection of all subsets of \( X \) that are Carathéodory measurable with respect to \( \mu^* \) and let \( \mu : \mathcal{B} \rightarrow [0, \infty] \) be the restriction of \( \mu^* \) to \( \mathcal{B} \). Then \( \mathcal{B} \) is a \( \sigma \)-algebra and \( \mu \) is a measure.
Proof: The empty set lies in \(B\) and so does the complement of every set in \(B\). Let \(S, T \in B\) and \(A \subset X\). Decompose \(A\) into four disjoint sets:

\[
A_{00} = A \setminus (S \cup T) \quad A_{10} = S \cap (A \setminus T) \quad A_{01} = T \cap (A \setminus S) \quad A_{11} = A \cap S \cap T
\]

The Carathéodory measurability of \(S\) establishes that

\[
\mu^*(A) = \mu^*(A_{00} \cup A_{10} \cup A_{11} \cup A_{01}) = \mu^*(A_{10} \cup A_{11}) + \mu^*(A_{00} \cup A_{01})
\]

and

\[
\mu^*(A \cap (S \cup T)) = \mu^*(A_{10} \cup A_{11} \cup A_{01}) = \mu^*(A_{10} \cup A_{11}) + \mu^*(A_{01})
\]

The Carathéodory measurability of \(T\) guarantees that

\[
\mu^*(A \setminus S) = \mu^*(A_{00} \cup A_{01}) = \mu^*(A_{01}) + \mu^*(A_{00})
\]

It then follows that \(S \cup T \in B\):

\[
\mu^*(A) = \mu^*(A_{10} \cup A_{11}) + \mu^*(A_{01}) + \mu^*(A_{00}) = \mu^*(A \cap (S \cup T)) + \mu^*(A \setminus (S \cup T)).
\]

This establishes that \(B\) is a Boolean algebra. In view of Problem 1.56, to show that \(B\) is in fact a \(\sigma\)-algebra we only need to establish that it is closed under countable disjoint unions. Let \(S_1, S_2, \ldots\) be a sequence of disjoint sets in \(B\) and let \(A \subset X\). Let \(U_n = \bigcup_{k=1}^{n} S_k\) for \(n \in \mathbb{N}\) and \(n = \infty\). For each \(n \in \mathbb{N}\), we have already shown that \(U_n \in B\) and hence

\[
\mu^*(A) = \mu^*(A \cap U_n) + \mu^*(A \setminus U_n)
\]

and

\[
\mu^*(A \cap U_{n+1}) = \mu^*(A \cap U_n) + \mu^*(A \cap S_{n+1})
\]

using the fact that \(U_{n+1} \supset U_n\) and disjointness. Using induction and countable subadditivity, we then see that

\[
\lim_{n \to \infty} \mu^*(A \cap U_n) = \sum_{k=1}^{\infty} \mu^*(A \cap S_k) \geq \mu^*(A \cap U_\infty). \quad (1.1)
\]

Moreover, we know from monotonicity that

\[
\mu^*(A \setminus U_n) \geq \mu^*(A \setminus U_\infty)
\]

for each \(n \in \mathbb{N}\) and hence

\[
\lim_{n \to \infty} \mu^*(A \setminus U_n) \geq \mu^*(A \setminus U_\infty)
\]

On combining these results, we obtain

\[
\mu^*(A) \geq \mu^*(A \cap U_\infty) + \mu^*(A \setminus U_\infty).
\]
It then follows from subadditivity that $U_\infty \in \mathcal{B}$.

Finally, on setting $A = X$ in Eq. (1.1), we see that

$$
\lim_{n \to \infty} \mu^*(U_n) = \sum_{k=1}^{\infty} \mu^*(S_k).
$$

But from monotonicity and subadditivity, we know for each $n \in \mathbb{N}$ that

$$
\mu^*(U_n) \leq \mu^*(U_\infty) \leq \sum_{k=1}^{\infty} \mu^*(S_k).
$$

On taking the limit as $n \to \infty$, we see that

$$
\mu^*(U_\infty) = \sum_{k=1}^{\infty} \mu^*(S_k).
$$

That is, the restriction $\mu$ of $\mu^*$ to $\mathcal{B}$ is in fact a measure.

Q. We have seen that the finitely additive elementary measure can be extended to the countably additive Lebesgue measure. Given a finitely additive measure $\mu_0$ on a Boolean algebra $\mathcal{B}_0$, is it always possible to find a $\sigma$-algebra $\mathcal{B}$ refining $\mathcal{B}_0$ and a countably additive measure $\mu$ that is the extension of $\mu_0$ to $\mathcal{B}$?

A. An obvious necessary condition for this is that $\mu_0$ is countably additive on $\mathcal{B}_0$. It turns out that this condition is also sufficient.

Definition: A **pre-measure** on a Boolean algebra $\mathcal{B}_0$ is a finitely additive measure $\mu_0 : \mathcal{B}_0 \to [0, \infty]$ such that $\mu_0(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu_0(E_k)$ whenever $E_1, E_2, \ldots$ are disjoint subsets of $\mathcal{B}_0$ such that $\bigcup_{k=1}^{\infty} E_k \in \mathcal{B}_0$.

**Theorem 1.24** (Hahn–Kolmogorov): Hahn–Kolmogorov Every pre-measure $\mu_0 : \mathcal{B}_0 \to [0, \infty]$ on a Boolean algebra $\mathcal{B}_0$ in $X$ can be extended to a countably additive measure $\mu : \mathcal{B} \to [0, \infty]$.

Proof: In analogy with the how elementary measure (specifically the volume of boxes) was used to construct the Lebesgue outer measure, we define for any subset $A$ of $X$

$$
\mu^*(A) = \inf_{\bigcup_{k=1}^{\infty} E_k \supseteq A} \sum_{k=1}^{\infty} \mu_0(E_k).
$$

which, following the proof of Theorem 1.2, we see is an outer measure.

Let $\mathcal{B}$ be the collection of all subsets of $X$ that are Carathéodory measurable with respect to $\mu^*$. By Theorem 1.23, we know that $\mathcal{B}$ is a $\sigma$-algebra and the restriction $\mu$
of \( \mu^* \) to \( B \) is a countably additive measure, so we only need to show that \( B \) contains \( B_0 \) and \( \mu \) extends \( \mu_0 \).

Let \( \epsilon > 0 \). Given \( E \subset B_0 \) and \( A \subset X \), we can find sets \( E_1, E_2, \ldots \in B_0 \) that cover \( A \) such that
\[
\sum_{k=1}^{\infty} \mu_0(E_k) < \mu^*(A) + \epsilon.
\]
Since \( \mu_0 \) is a pre-measure on \( B_0 \), we know for each \( n \in \mathbb{N} \) that
\[
\mu_0(E_n \cap E) + \mu_0(E_n \setminus E) = \mu_0(E_n).
\]
Since the sets \( E_n \cap E \) lie in \( B_0 \) and cover \( A \cap E \), we know that
\[
\mu^*(A \cap E) \leq \sum_{n=1}^{\infty} \mu_0(E_n \cap E).
\]
Likewise,
\[
\mu^*(A \setminus E) \leq \sum_{n=1}^{\infty} \mu_0(E_n \setminus E).
\]
Hence
\[
\mu^*(A \cap E) + \mu^*(A \setminus E) \leq \sum_{n=1}^{\infty} \mu_0(E_n) \leq \mu^*(A) + \epsilon.
\]
Since \( \epsilon \) is arbitrary, we then see that
\[
\mu^*(A \cap E) + \mu^*(A \setminus E) \leq \mu^*(A) + \epsilon.
\]
Since subadditivity implies the reverse inequality, we see that
\[
\mu^*(A \cap E) + \mu^*(A \setminus E) = \mu^*(A).
\]
That is, \( E \in B \).

Since each subset \( E \) of \( B_0 \) covers itself, \( \mu^*(E) \leq \mu_0(E) \). From each covering \( E_1, E_2, \ldots \in B_0 \) of \( E \), construct the disjoint subsets \( E \cap E_n \setminus \bigcup_{k=1}^{n-1} E_k \) of \( B_0 \), whose union is exactly equal to \( E \). Since \( \mu_0 \) is a pre-measure we then know, using monotonicity, that
\[
\mu_0(E) = \sum_{n=1}^{\infty} \mu_0\left(E \cap E_n \setminus \bigcup_{k=1}^{n-1} E_k\right) \leq \sum_{n=1}^{\infty} \mu_0(E_n).
\]
This provides us with a lower bound for \( \mu^*(E) \):
\[
\mu^*(E) \geq \mu_0(E),
\]
so that \( \mu^*(E) = \mu_0(E) \). We thus see that the restriction \( \mu \) of \( \mu^* \) to \( B \) extends \( \mu_0 \) from the Boolean algebra \( B_0 \) to the \( \sigma \)-algebra \( B \).
**Definition:** A measure space \((X, \mathcal{B}, \mu)\) is \(\sigma\)-finite if \(X\) can be expressed as the countable union of sets of finite measure.

- \(\mathbb{R}^d\) with the Lebesgue measure is \(\sigma\)-finite.

- \(\mathbb{R}^d\) with the counting measure is not \(\sigma\)-finite.

**Problem 1.57:** Let \(\mu_0 : \mathcal{B}_0 \to [0, \infty]\) be a \(\sigma\)-finite pre-measure on a Boolean algebra \(\mathcal{B}_0\) in a space \(X\). Suppose that \(\mu, \mu' : \mathcal{B} \to [0, \infty]\) are Hahn–Kolmogorov extensions of \(\mu_0\) to \(\sigma\)-algebras \(\mathcal{B}\) and \(\mathcal{B}'\) containing \(\mathcal{B}_0\). Show that \(\mu = \mu'\) on \(\mathcal{B} \cap \mathcal{B}'\).

**Remark:** We now introduce a powerful tool for constructing measures.

**Theorem 1.25** (Lebesgue-Stieltjes measure): Let \(F : \mathbb{R} \to \mathbb{R}\) be a monotone increasing function and define

\[
F_-(x) \doteq \sup_{y < x} F(y), \quad F_+(x) \doteq \inf_{y > x} F(y),
\]

so that \(F_-(x) \leq F(x) \leq F_+(x)\) for all \(x \in \mathbb{R}\). Then there exists a unique Borel measure \(\mu_F : \mathcal{B}[\mathbb{R}] \to [0, \infty]\), known as the Lebesgue-Stieltjes measure of \(F\), such that

\[
\begin{align*}
\mu_F([a, b]) &= F_+(b) - F_-(a), \\
\mu_F((a, b]) &= F_+(b) - F_-(a), \\
\mu_F((a, b)) &= F_+(b) - F_+(a), \\
\mu_F((a, b)) &= F_-(b) - F_+(a),
\end{align*}
\]

whenever \(-\infty < a < b < \infty\) and

\[
\mu_F([a, a]) = F_+(a) - F_-(a).
\]

**Proof:** See Tao, Theorem 1.7.9.

- When \(F(x) = x\), \(\mu_F\) is just the Lebesgue measure \(m\).
• If \( F : \mathbb{R} \to \mathbb{R} \) is increasing and absolutely continuous (so that, by Theorem 1.21, \( F' \) exists almost everywhere and is absolutely integrable on every closed interval \([a, b] \)),

\[
\mu_F(S) = \int_S F'(x) \, dx
\]

for any Borel-measurable set \( S \). Then for any unsigned Borel-measurable function \( f : \mathbb{R} \to [0, \infty] \), one can define the Lebesgue–Stieltjes integral of \( f \) with respect to \( F \):

\[
\int_{\mathbb{R}} f \, d\mu_F = \int_{\mathbb{R}} f(x) F'(x) \, dx.
\]

The Lebesgue–Stieltjes integral is often abbreviated as \( \int f \, dF \). In particular when \( f = 1 \) we obtain another formulation of the fundamental theorem of calculus:

\[
\int_{[a,b]} dF = F_+(b) - F_-(a),
\]

whenever \(-\infty < a < b < \infty\).

**Definition:** A Radon measure on \( \mathbb{R} \) is a Borel measure \( \mu \) obeying:

1. (Local finiteness) \( \mu(K) < \infty \) for every compact \( K \);
2. (Inner regularity) One has \( \mu(S) = \sup_{K \subseteq E} \mu(K) \) for every Borel set \( S \);
3. (Outer regularity) One has \( \mu(S) = \inf_{U \supseteq E} \mu(U) \) for every Borel set \( S \).

**Problem 1.58:** Show that the Lebesgue–Stieltjes measure \( \mu_F \) for every monotone function \( F : \mathbb{R} \to \mathbb{R} \) is a Radon measure on \( \mathbb{R} \). Conversely, if \( \mu \) is a Radon measure on \( \mathbb{R} \), show that there exists a monotone function \( F : \mathbb{R} \to \mathbb{R} \) such that \( \mu = \mu_F \).

**Definition:** Let \((X, \mathcal{B}_X)\) and \((Y, \mathcal{B}_Y)\) be measurable spaces. The \( \sigma \)-algebra

\[
\mathcal{B}_X \times \mathcal{B}_Y = \{ \{ A \times B : A \in \mathcal{B}_X, B \in \mathcal{B}_Y \} \},
\]

in \( X \times Y \) is a product \( \sigma \)-algebra.

**Remark:** The \( \sigma \)-algebra \( \mathcal{B}_X \times \mathcal{B}_Y \) is the coarsest \( \sigma \)-algebra on \( X \times Y \) for which every product of a \( \mathcal{B}_X \)-measurable set and a \( \mathcal{B}_Y \)-measurable set is \( \mathcal{B}_X \times \mathcal{B}_Y \) measurable.
Remark: If $f : X \times Y \to [0, \infty]$ is measurable with respect to $\mathcal{B}_X \times \mathcal{B}_Y$, the function $f_x : y \mapsto f(x, y)$ is $\mathcal{B}_Y$ measurable for every $x \in X$ and $f^y : x \mapsto f(x, y)$ is $\mathcal{B}_X$ measurable for every $y \in Y$.

Theorem 1.26 (Existence and uniqueness of product measure): Let $(X, \mathcal{B}_X, \mu_X)$ and $(Y, \mathcal{B}_Y, \mu_Y)$ be $\sigma$-finite measure spaces. Then there exists a unique product measure $\mu_{X \times Y}$ on $\mathcal{B}_X \times \mathcal{B}_Y$ such that 
\[(\mu_{X \times Y})(E \times F) = \mu_X(E) \mu_Y(F)\]
whenever $E \in \mathcal{B}_X$ and $F \in \mathcal{B}_Y$.

Proof: Consider the Boolean algebra $\mathcal{B}_0$ consisting of all finite unions
\[S = (E_1 \times F_1) \cup \ldots \cup (E_m \times F_m)\]
of Cartesian products of $\mathcal{B}_x$ measurable sets $E_1, \ldots, E_m$ and $\mathcal{B}_y$ measurable sets $F_1, \ldots, F_m$. In analogy with the decomposition of elementary sets in $\mathbb{R}^d$ into finite unions of disjoint boxes, we can without loss of generality assume that $E_1 \times F_1, \ldots, E_m \times F_m$ are disjoint. We then introduce the finitely additive measure $\mu_0 : \mathcal{B}_0 \to [0, \infty]$
\[\mu_0(S) = \sum_{k=1}^{m} \mu_X(E_k) \mu_Y(F_k),\]
whenever $S$ is a disjoint union of Cartesian products $E_1 \times F_1, \ldots, E_m \times F_m$ of $\mathcal{B}_x$ and $\mathcal{B}_y$ measurable sets, independent of exactly how $S$ is decomposed.

To establish that $\mu_0$ is a pre-measure, we need to show for every countable union $S \in \mathcal{B}_0$ of disjoint sets $S_1, S_2, \ldots \in \mathcal{B}_0$ that $\mu_0(S) = \sum_{k=1}^{\infty} \mu_0(S_k)$. Using the finite additivity of $\mu_0$, we can reduce the problem to the case $m = 1$; that is, $S = E \times F$ and $S_k = E_k \times F_k$ for $k = 1, 2, \ldots$. We begin by re-expressing the decomposition $S = \bigcup_{k=1}^{\infty} S_k$ into disjoint sets as the pointwise identity
\[1_E(x) 1_F(y) = \sum_{k=1}^{\infty} 1_{E_k}(x) 1_{F_k}(y)\]
for all $x \in X$ and $y \in Y$. For fixed $x \in X$, we can integrate both sides in $y$:
\[\int_Y 1_E(x) 1_F(y) \, d\mu_Y(y) = \int_Y \sum_{k=1}^{\infty} 1_{E_k}(x) 1_{F_k}(y) \, d\mu_Y(y).\]
Using the monotone convergence theorem to interchange the integration and summation on the right-hand side, this identity evaluates to
\[1_E(x) \mu_Y(F) = \sum_{k=1}^{\infty} 1_{E_k}(x) \mu_Y(F_k).\]
Similarly, we may now integrate in $x$ and use monotone convergence to obtain
\[\mu_X(E) \mu_Y(F) = \sum_{k=1}^{\infty} \mu_X(E_k) \mu_Y(F_k);\]
thereby demonstrating that $\mu_0$ is a pre-measure.

Using the Hahn–Kolmogorov theorem, we can then extend $\mu_0$ to a countably additive measure $\mu_X \times \mu_Y$ on a $\sigma$-algebra containing $\mathcal{B}_0$. Since $\mathcal{B}_X \times \mathcal{B}_Y$ is the coarsest such $\sigma$-algebra, we see that $\mu_X \times \mu_Y$ is a countably additive measure on $\mathcal{B}_X \times \mathcal{B}_Y$, with $(\mu_X \times \mu_Y)(E \times F) = \mu_X(E)\mu_Y(F)$ whenever $E \in \mathcal{B}_X$ and $F \in \mathcal{B}_Y$. The uniqueness of $\mu_X \times \mu_Y$ follows from Prob 1.57.

**Definition:** A monotone class in a space $X$ is a collection $\mathcal{B}$ of subsets of $X$ such that

1. The union $\bigcup_{k=1}^{\infty} S_k$ of every countable increasing sequence of sets $S_1 \subset S_2 \subset \ldots$ in $\mathcal{B}$ is itself in $\mathcal{B}$.

2. The intersection $\bigcap_{k=1}^{\infty} S_k$ of every countable decreasing sequence of sets $S_1 \supset S_2 \supset \ldots$ in $\mathcal{B}$ is itself in $\mathcal{B}$.

**Remark:** Every $\sigma$-algebra is a monotone class, but a monotone class need not be a $\sigma$-algebra.

**Lemma 1.19 (Monotone class lemma):** Let $\mathcal{B}$ be a Boolean algebra on $X$. Then $\langle \mathcal{B} \rangle$ is the smallest monotone class containing $\mathcal{B}$.

**Proof:** Let $\mathcal{I}$ be the intersection of all monotone classes containing $\mathcal{B}$. Since $\langle \mathcal{I} \rangle$ is one such class, we see that $\mathcal{I} \subset \langle \mathcal{B} \rangle$.

Consider the set $\mathcal{J} \subset \mathcal{I}$ composed of those elements of $\mathcal{I}$ whose complements are also in $\mathcal{I}$. Since $(\bigcup_{k=1}^{\infty} S_k)^c = \bigcap_{k=1}^{\infty} S_k^c$, we see that $\mathcal{J}$ is also a monotone class containing $\mathcal{B}$, so that $\mathcal{I} \subset \mathcal{J}$. Hence $\mathcal{J} = \mathcal{I}$ is closed under complements.

Likewise, consider the set $\mathcal{K} \subset \mathcal{I}$ composed of those elements of $\mathcal{I}$ for which $S \setminus T$, $T \setminus S$, $S \cap T$, and $X \setminus (S \cup T)$ also belong to $\mathcal{I}$ for every $T \in \mathcal{I}$. Since $(\bigcup_{k=1}^{\infty} S_k) \setminus T = \bigcup_{k=1}^{\infty} (S_k \setminus T)$, etc., we see that $\mathcal{K}$ is also a monotone class containing $\mathcal{B}$, so that $\mathcal{I} \subset \mathcal{K}$. Thus $\mathcal{I} = \mathcal{K}$.

We now know that $\mathcal{I}$ is closed under finite unions as well as finite intersections. Since $\mathcal{I} \supset \mathcal{B}$ and $\emptyset \in \mathcal{B}$, we see thus see that $\mathcal{I}$ is a Boolean algebra. Given any countable sequence of subsets $S_k$ of $\mathcal{I}$, $k = 1, 2, \ldots$, consider the increasing sequence $T_n = \bigcup_{k=1}^{n} S_k$ for $n = 1, 2, \ldots$. Since $\mathcal{I}$ is a monotone class, we know that $\bigcup_{k=1}^{\infty} S_k = \bigcup_{n=1}^{\infty} T_n \in \mathcal{I}$, so $\mathcal{I}$ is a $\sigma$-algebra. Since $\mathcal{I}$ contains $\mathcal{B}$, we deduce that $\langle \mathcal{I} \rangle \supset \langle \mathcal{B} \rangle$. Thus $\langle \mathcal{I} \rangle = \langle \mathcal{B} \rangle$.

**Theorem 1.27 (Tonelli’s theorem):** Let $(X, \mathcal{B}_X, \mu_X)$ and $(Y, \mathcal{B}_Y, \mu_Y)$ be $\sigma$-finite measure spaces and let $f : X \times Y \to [0, \infty]$ be measurable with respect to $\mathcal{B}_X \times \mathcal{B}_Y$. Then:
The functions \( x \rightarrow \int_Y f(x,y) \, d\mu_Y(y) \) and \( y \rightarrow \int_X f(x,y) \, d\mu_X(x) \) are measurable with respect to \( \mathcal{B}_X \) and \( \mathcal{B}_Y \), respectively;

\[
\int_{X \times Y} f(x,y) \, d\mu_X \times \mu_Y(x,y) = \int_X \int_Y f(x,y) \, d\mu_Y(y) \, d\mu_X(x) = \int_Y \int_X f(x,y) \, d\mu_X(x) \, d\mu_Y(y).
\]

Proof: Since the \( \sigma \)-finite spaces \( X \) and \( Y \) can be decomposed as increasing unions of finite-measure sets, the monotone convergence theorem monotone convergence can be used to reduce the problem to the case where \( X \) and \( Y \), and by Theorem 1.26, \( X \times Y \), have finite measure.

Since every unsigned measurable function is the limit of an increasing sequence of unsigned simple functions, by the monotone convergence theorem and linearity, we can further reduce the problem to the case where \( f = 1_S \) for some \( S \in \mathcal{B}_X \times \mathcal{B}_Y \).

Let \( \mathcal{C} \) be the collection of all subsets of \( \mathcal{B}_X \times \mathcal{B}_Y \) for which the claim holds. By repeated application of the monotone convergence theorem and downward monotone convergence (which applies in this finite measure setting), we see that \( \mathcal{C} \) is a monotone class. Theorem 1.26 establishes that every product \( E \times F \) with \( E \subset \mathcal{B}_X \) and \( F \subset \mathcal{B}_Y \) is in \( \mathcal{C} \). By finite additivity, every finite union

\[ (E_1 \times F_1) \cup \ldots \cup (E_m \times F_m) \]

of such products, which can always be re-expressed as a union of disjoint products, is an element of \( \mathcal{C} \). Since \( \mathcal{C} \) is a monotone class, Lemma 1.19 then establishes that \( \mathcal{C} \) contains the entire \( \sigma \)-algebra \( \mathcal{B}_X \times \mathcal{B}_Y \).

Corollary 1.27.1: Let \( (X, \mathcal{B}_X, \mu_X) \) and \( (Y, \mathcal{B}_Y, \mu_Y) \) be \( \sigma \)-finite measure spaces and let \( E \in \mathcal{B}_X \times \mathcal{B}_Y \) be a null set with respect to \( \mu_X \times \mu_Y \). Then for \( \mu_X \)-almost every \( x \in X \) the set \( \{ E_x = \{ y \in Y : (x,y) \in E \} \} \) is a \( \mu_Y \)-null set. Similarly, for \( \mu_Y \)-almost every \( y \in Y \) the set \( \{ E^y = \{ x \in X : (x,y) \in E \} \} \) is a \( \mu_X \)-null set.

Proof: On applying Tonelli’s theorem to the indicator function \( 1_E \), we find

\[
0 = \int_X \int_Y 1_E(x,y) \, d\mu_Y(y) \, d\mu_X(x) = \int_Y \int_X 1_E(x,y) \, d\mu_X(x) \, d\mu_Y(y).
\]

Thus

\[
0 = \int_X \mu_Y(E_x) \, d\mu_X(x) = \int_Y \mu_X(E^y) \, d\mu_Y(y),
\]

from which the result immediately follows.

Remark: One limitation of the product \( \sigma \)-algebra \( \mathcal{B}_X \times \mathcal{B}_Y \) is that it may not be complete.
Consider the two-dimensional Cartesian product of the one-dimensional Lebesgue measure space \((\mathbb{R}, \mathcal{L}[\mathbb{R}], m)\) with itself. This yields a measure \(m^2\) on \(\mathbb{R}^2\) that is incomplete. For example, let \(V\) be a nonmeasurable set like the Vitali set constructed on p. 9 and consider \(V \times \{0\}\) is not measurable in \(\mathbb{R}^2\) even though \(V \times \{0\}\) is a subset of the \(m^2\)-null set \(\mathbb{R} \times \{0\}\). We can fix this deficiency by replacing \(\mathbb{R}^2\) by its completion \(\overline{\mathbb{R}^2}\).

**Remark:** It is easy to extend Tonelli’s theorem to complete product measures.

**Corollary 1.27.2** (Tonelli’s theorem for complete product measures): Let \((X, \mathcal{B}_X, \mu_X)\) and \((Y, \mathcal{B}_Y, \mu_Y)\) be complete \(\sigma\)-finite measure spaces and let \(f : X \times Y \to [0, \infty]\) be measurable with respect to \(\mathcal{B}_X \times \mathcal{B}_Y\). Then:

(i) The \(\mu_X\)-almost everywhere defined function \(x \to \int_Y f(x, y) \, d\mu_Y(y)\) and \(\mu_Y\)-almost everywhere defined function \(y \to \int_X f(x, y) \, d\mu_X(x)\) are measurable with respect to \(\mathcal{B}_X\) and \(\mathcal{B}_Y\), respectively;

(ii) 
\[
\int_{X \times Y} f(x, y) \, d\mu_X \times \mu_Y(x, y) = \int_X \int_Y f(x, y) \, d\mu_Y(y) \, d\mu_X(x) = \int_Y \int_X f(x, y) \, d\mu_X(x) \, d\mu_Y(y).
\]

Proof: Every measurable set in \(\overline{\mathcal{B}_X \times \mathcal{B}_Y}\) is equal to a measurable set in \(\mathcal{B}_X \times \mathcal{B}_Y\) outside of a \(\mu_X \times \mu_Y\)-subnull set. The result then follows from Theorem 1.27 and Corollary 1.27.1.

**Corollary 1.27.3** (Fubini’s theorem): Let \((X, \mathcal{B}_X, \mu_X)\) and \((Y, \mathcal{B}_Y, \mu_Y)\) be complete \(\sigma\)-finite measure spaces and let \(f : X \times Y \to \mathbb{C}\) be absolutely integrable with respect to \(\overline{\mathcal{B}_X \times \mathcal{B}_Y}\). Then:

(i) The \(\mu_X\)-almost everywhere defined function \(x \to \int_Y f(x, y) \, d\mu_Y(y)\) and \(\mu_Y\)-almost everywhere defined function \(y \to \int_X f(x, y) \, d\mu_X(x)\) are absolutely integrable with respect to \(\mu_X\) and \(\mu_Y\), respectively;

(ii) 
\[
\int_{X \times Y} f(x, y) \, d\mu_X \times \mu_Y(x, y) = \int_X \int_Y f(x, y) \, d\mu_Y(y) \, d\mu_X(x) = \int_Y \int_X f(x, y) \, d\mu_X(x) \, d\mu_Y(y).
\]

Proof: The result follows on applying Theorem 1.27.2 to the positive and negative parts of the real and imaginary parts of \(f\), noting from the absolute integrability of \(f\) that \(\int_Y f(x, y) \, d\mu_Y(y)\) is finite for \(\mu_X\)-almost all \(x \in X\) and that \(\int_X f(x, y) \, d\mu_X(x) \, d\mu_Y(y)\) is finite for \(\mu_Y\)-almost all \(y \in Y\).
Remark: If \( \int_X \int_Y |f(x,y)| \, d\mu_Y(y) \, d\mu_X(x) < \infty \), Tonelli’s theorem can be used to establish the absolute integrability requirement of the Fubini theorem.

Remark: The Fubini theorem provides us with a practical tool for computing integrals.

**Problem 1.59:** (Area interpretation of integral) Let \((X, \mathcal{B}, \mu)\) be a \(\sigma\)-finite measure space and let \(\mathbb{R}\) be equipped with the Lebesgue measure \(m\) and the Borel \(\sigma\)-algebra \(\mathcal{B}[\mathbb{R}]\). Show that \(f : X \to [0, \infty]\) is measurable iff \(S = \{(x, t) \in X \times \mathbb{R} : 0 \leq t \leq f(x)\}\) is measurable in \(\mathcal{B} \times \mathcal{B}[\mathbb{R}]\), with

\[
(\mu \times m)(S) = \int_X f(x) \, d\mu(x).
\]

Since \(X\) is \(\sigma\)-finite, we can project \(X \times \mathbb{R}\) onto a countable sequence of finite-measure sets, on which the indicator function \(1_S\) is absolutely integrability \(\iff\) \(f\) is measurable, noting that the interval \([0, f(x)]\) is always Lebesgue measurable. We then apply the Fubini theorem. Monotone convergence yields

\[
(\mu \times m)(S) = \int_{X \times \mathbb{R}} 1_S \, d\mu \times m = \int_X \int_{\mathbb{R}} 1_{[0, f(x)]} \, dt \, d\mu(x) = \int_X f(x) \, d\mu(x).
\]

**Problem 1.60:** (Distribution formula). Let \((X, \mathcal{B}, \mu)\) be a \(\sigma\)-finite measure space, and let \(f : X \to [0, \infty]\) be measurable. Show that

\[
\int_X f(x) \, d\mu = \int_{[0, \infty]} \mu(\{x \in X : f(x) \geq t\}) \, dt.
\]

For each \(t\), let \(S_t = \{x \in X : f(x) \geq t\}\). First note that \(\mu(S_t)\) is a monotone function in \(t\) and monotone functions are Lebesgue measurable (since their inverse images are intervals). Following Prob 1.59, we then see that

\[
\int_{[0, \infty]} \mu(S_t) \, dt = \int_{[0, \infty]} \int_X 1_{S_t} \, d\mu(x) \, dt = \int_X \int_{[0, \infty]} 1_{[0, f(x)]} \, dt \, d\mu(x) = \int_X f(x) \, d\mu(x).
\]
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