MATH 417 Section Q1
Final Exam

Dr. J. Bowman
Assigned 18:30 08 April 2020
Due 11:59 13 April 2020

Name (Last, First): ________________________________

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• This is an open book exam. If anything is unclear, please ask your instructor! You are not allowed to work with or ask for help from anyone else. You may consult your lecture notes and textbooks. You may not use without proof any result that we have not covered in class, on the assignments, or midterm examination.

• Check that you have 2 pages.

• This exam consists of 6 questions, for a total of 21 points.
1. Evaluate the Lebesgue–Stieltjes integral
   (a) \[ \int_1^e \frac{d(\log x)}{1 + x^2} \]
   (b) \[ \int_{-1}^1 x^2 \, d|x| \]

2. Let \( A \subset \mathbb{R}^d \). Show that there exists a \( G_\delta \)-subset \( G \) of \( \mathbb{R}^d \) such that \( A \subset G \) and \( m^*(G) = m^*(A) \).

3. Let \( S \) be a Lebesgue measurable set.
   (a) Show that there exists a Borel-measurable set \( B \supset S \) with \( m(B) = m(S) \).
   (b) Show that there exists a null Borel set \( C \supset B \setminus S \).
   (c) Use parts (a) and (b) to prove that the Lebesgue \( \sigma \)-algebra on \( \mathbb{R}^d \) is generated by the union of the Borel \( \sigma \)-algebra and the null \( \sigma \)-algebra.

4. Let \( f : [a, b] \to \mathbb{R} \) be continuous, and let \( \alpha : [a, b] \to \mathbb{R} \) be a function of bounded variation. Show that the function
   \[ F_\alpha : [a, b] \to \mathbb{R}, \quad x \mapsto \int_a^x f(t) \, d\alpha(t) \]
   is also of bounded variation. (Hint: Consider first the case where \( \alpha \) is increasing.)

5. Let \( X = \mathbb{N} \) and \( \mu \) be the counting measure on \( X \). Let \( Y = (0, \infty) \) and \( \nu \) be a Borel measure on \( Y \) satisfying \( \nu((t, \infty)) = e^{-t^2} \) for every \( t > 0 \). Let \( g : X \times Y \to \mathbb{R} \) be defined by \( g(x, y) = y^2/3^x \). Evaluate
   \[ \int_{X \times Y} g \, d\mu \times \nu. \]

6. Let \( (X, \mathcal{B}, \mu) \) be a \( \sigma \)-finite measure space and let \( \mathbb{R} \) be equipped with the Lebesgue measure \( m \) and the Borel \( \sigma \)-algebra \( \mathcal{B}[\mathbb{R}] \). Show that the graph \( G = \{(x, f(x) : x \in X \} \) of a measurable function \( f : X \to [0, \infty] \) is a \( \mu \times m \)-null set.