1. Let \( f : \mathbb{R} \to \mathbb{R} \) be a strictly increasing continuous function. Consider the collection \( \mathcal{B} \) of sets \( S \subset \mathbb{R} \) whose images \( f(S) \) are Borel sets on \( \mathbb{R} \).

(a) Prove that \( \mathcal{B} \) is a \( \sigma \)-algebra on \( \mathbb{R} \).

First, \( f(\emptyset) = \emptyset \) is a Borel set. Moreover, \( f \) is injective, so \( f(S^c) = f(\mathbb{R}) \setminus f(S) \), which is a Borel set since \( f(\mathbb{R}) \) is an interval. Finally, \( f(\bigcup_{k=1}^{\infty} S_k) = \bigcup_{k=1}^{\infty} f(S_k) \in \mathcal{B} \).

(b) Let \( A \subset \mathbb{R} \) be a Borel set. Prove that \( f(A) \) is also a Borel set.

Note that \( f \) has a continuous inverse \( f^{-1} \) that pulls back the open set \( S \) to an open set \( f(S) \) (alternatively, since the continuous strictly increasing function \( f \) maps open intervals to open intervals, it also maps open sets to open sets.) Thus \( \mathcal{B} \) contains all open sets. Part (a) then implies that \( \mathcal{B} \) contains the entire Borel \( \sigma \)-algebra. In particular, since \( A \) is a Borel set, \( A \in \mathcal{B} \); that is, \( f(A) \) is a Borel set.

2. Let \((X, \mathcal{B})\) be a measurable space. Show that an unsigned function \( f \) is measurable \( \Longleftrightarrow \) for every rational \( r \) the set \( \{x \in X : f(x) > r\} \) is \( \mathcal{B} \)-measurable.

\( \Rightarrow \) This follows immediately from Prob 3.8.

\( \Leftarrow \) This also follows from Prob 3.8 since for real \( \lambda \)

\[
\{x \in X : f(x) > \lambda\} = \bigcup_{r \in \mathbb{Q} : r > \lambda} \{x \in X : f(x) > r\}.
\]
3. Consider the Cantor function $F$.

(a) Show that $F(C^c)$ is countable, where $C^c$ denotes the complement of the Cantor set in $[0, 1]$.

This follows from the fact that $C^c$ is a countable union of intervals on which $F$ is locally constant; $F$ therefore achieves only a countable number of values on $C^c$.

(b) Show that $m(F(C)) = 1$.

Since $F$ is continuous and $F[0] = 0$ and $F[1] = 1$, we know from the intermediate value theorem that

$$[0, 1] = F([0, 1]) = F(C \cup C^c) = F(C) \cup F(C^c),$$

From part (a) we know that $F(C^c)$ is a Lebesgue null set and hence, using the completeness of the Lebesgue measure, so is $[0, 1] \setminus F(C) \subset F(C^c)$. We thus see that $F(C) \setminus \{0, 1\}$ is Lebesgue measurable and hence so is $F(C)$. Subadditivity and monotonicity then imply that

$$1 = m([0, 1]) \leq m(F(C)) + m(F(C^c)) = m(F(C)) \leq m([0, 1]) = 1.$$

**Alternative solution**: since $C$ is closed and $F$ is continuous, $F(C)$ contains its limit points, including $F(C^c)$. So $F(C) = [0, 1]$.

(c) Show that $G : x \mapsto F(x) + x$ is strictly monotonic and continuous on $[0, 1]$, so that it has a continuous inverse $G^{-1}$.

Since $G$ is the sum of two continuous functions, it is continuous. Moreover $x < y \Rightarrow F(x) + x < F(y) + y$, so $G$ is strictly increasing on $[0, 1]$.

(d) Show that $m(G(C)) = 1$.

Since $G$ maps each subinterval of $C^c$ to an interval of the same length, shifted by a constant, we know that $m(G(C^c)) = 1$. Being a countable union of intervals, $G(C^c)$ is Lebesgue measurable. Since $G$ is continuous and $G[0] = 0$ and $G[1] = 2$, we know from the intermediate value theorem that

$$[0, 2] = G([0, 1]) = G(C \cup C^c) = G(C) \cup G(C^c),$$

where $G(C) = [0, 2] \setminus G(C^c)$ and $G(C^c)$ are disjoint. Thus

$$m(G(C)) = 2 - m(G(C^c)) = 2 - 1 = 1.$$

(e) We have learned that one can construct a non-Lebesgue measurable subset of any set with positive Lebesgue measure. Let $S$ be a non-Lebesgue measurable subset of $G(C)$. Is $G^{-1}(S)$ Lebesgue measurable?

Yes, since $G^{-1}(S) \subset C$ and $m(C) = 0$, we know from the completeness of the Lebesgue $\sigma$-algebra that $G^{-1}(S)$ is a Lebesgue null set.
(f) Is \( G^{-1}(S) \) in part (e) Borel measurable?

Since \( G \) is strictly increasing and continuous on \([0, 1]\), we know from Question 1 that it maps Borel sets to Borel sets, which are Lebesgue measurable. So \( G^{-1}(S) \) cannot be a Borel set. We have thus constructed a (null and hence Lebesgue measurable) subset \( S \) of the Cantor set \( C \) that is not Borel measurable. Moreover, we see that the preimage \( S \) of the measurable function \( G^{-1} \) of the Lebesgue measurable set \( G^{-1}(S) \) is not Lebesgue measurable!

4. Let \( F : [a, b] \to \mathbb{R} \) be continuous. If \( F'(x) \) exists everywhere in \((a, b)\) and is bounded on \((a, b)\), is it always true that

\[
\int_{[a, b]} F' = F(b) - F(a)?
\]

Prove or provide a counterexample.

There exists \( B > 0 \) such that \( |F'(x)| \leq B \) for all \( x \in (a, b) \). Given \( \epsilon > 0 \), choose \( \delta = \epsilon/B \). Let \((a_1, b_1), \ldots, (a_n, b_n)\) be disjoint intervals with \( \sum_{k=1}^{n} (b_k - a_k) < \delta \). By the mean value theorem, there exists \( c_k \in (a_k, b_k) \) such that

\[
\sum_{k=1}^{n} |F(b_k) - F(a_k)| = \sum_{k=1}^{n} |F'(c_k)|(b_k - a_k) \leq B \sum_{k=1}^{n} (b_k - a_k) < B\delta = \epsilon.
\]

Thus, \( F \) is absolutely continuous and Theorem 5.7 guarantees that

\[
\int_{[a, b]} F' = F(b) - F(a).
\]

5. Define

\[
\mu^* : \mathcal{P}(\mathbb{R}) \to [0, \infty], \quad A \mapsto \begin{cases} 
0, & \text{if } A = \emptyset, \\
1, & \text{if } A \neq \emptyset \text{ is bounded}, \\
\infty, & \text{otherwise}.
\end{cases}
\]

(a) Show that \( \mu^* \) is an outer measure.

First, \( \mu^*(\emptyset) = 0 \).

Suppose \( A \subset B \neq \emptyset \). If \( A \) is bounded then \( \mu^*(A) \leq 1 \leq \mu^*(B) \). If \( A \) is unbounded, then so is \( B \): \( \mu^*(A) = \infty = \mu^*(B) \).

Let \( A_1, A_2, \ldots \subset \mathbb{R} \). We need to show

\[
\mu^* \left( \bigcup_{k=1}^{\infty} A_k \right) \leq \sum_{k=1}^{\infty} \mu^*(A_k) \tag{1}
\]

We may assume that each \( A_k \) is bounded as otherwise, both sides evaluate to \( \infty \). If all of the sets \( A_k \) are empty, then both sides evaluate to \( 0 \). If exactly \( m \geq 1 \) of these sets are nonempty, the left-hand side is \( 1 \) and the right-hand-side is \( m \geq 1 \).

Thus, \( \mu^* \) is an outer measure.
(b) Find all sets that are Carathéodory measurable with respect to $\mu^*$. Let $S \subset \mathbb{R}$ and suppose that $S$ is neither $\mathbb{R}$ nor $\emptyset$. Choose $x \in S$, $y \in S^c$ and consider $A = \{x, y\}$. Then
\[
\mu^*(A) = 1 < 2 = \mu^*(A \cap S) + \mu^*(A \cap S^c),
\]
so that $S$ is not Carathéodory measurable. We conclude that only $\emptyset$ and $\mathbb{R}$ are Carathéodory measurable with respect to $\mu^*$.

6. Evaluate the Lebesgue–Stieltjes integral
\[
\int_0^\pi \sin x \, d(\cos x).
\]
The integral evaluates to
\[
-\int_0^\pi \sin^2 x \, dx = -\int_0^\pi \frac{1 - \cos 2x}{2} \, dx = -\frac{1}{2} \left[ \frac{1 - \sin 2x}{2} \right]_0^\pi = -\frac{\pi}{2}.
\]

7. Let $X = \mathbb{N}$ and $\mu$ be the counting measure on $X$. Let $Y = (0, \infty)$ and $\nu$ be a Borel measure on $Y$ satisfying $\nu((t, \infty)) = e^{-t^3}$ for every $t > 0$. Let $g : X \times Y \to \mathbb{R}$ be defined by $g(x, y) = y^3/4^x$. Evaluate
\[
\int_{X \times Y} g \, d(\mu \times \nu).
\]
We first note that the measure space $(X, \mathcal{P}(X), \mu)$ is $\sigma$-finite since $\mathbb{N}$ is countable and $(Y, \mathcal{B}[Y], \nu)$ is $\sigma$-finite since $\nu(Y) = 1$. Also, being continuous in $y$, we see that $g$ is measurable with respect to $\mu \times \nu$. By Tonelli’s Theorem,
\[
\int_{X \times Y} g \, d(\mu \times \nu) = \int_X \int_Y \frac{y^3}{4^x} \, d\nu \, d\mu = \sum_{i=1}^{\infty} \frac{1}{4^i} \cdot \int_Y y^3 \, d\nu.
\]
The infinite sum evaluates to $1/3$. In the remaining integral, $\nu$ is the Lebesgue-Stieltjes measure induced by the monotonic increasing function $t \mapsto -e^{-t^3}$. Hence,
\[
\int_Y y^3 \, d\nu = \int_0^{\infty} y^3 e^{-y^3} 3y^2 \, dy,
\]
which, on letting $u = y^3$, becomes
\[
\int_0^{\infty} u e^{-u} \, du = [-ue^{-u}]_0^{\infty} + \int_0^{\infty} e^{-u} \, du = 1.
\]
Hence, the integral is $1/3$. 

4