1. For each \( n \in \mathbb{N} \) let \( f_n : [0, 1] \to [0, \infty) \) be defined by 
\[
x \mapsto \begin{cases} 
  n & \text{if } x \in \left[ \frac{q}{2^p}, \frac{q+1}{2^p} \right], \text{ where } n = 2^p + q, \text{ for } p \in \mathbb{N} \cup \{0\}, q \in [0, 2^p), \\
  0 & \text{otherwise.}
\end{cases}
\]

Justify your answers to the following questions:

(a) Does \( f_n \to 0 \) in (Lebesgue) measure?

Yes, since for each \( \epsilon > 0 \)
\[
m\left( \{ x : |f_n| > \epsilon \} \right) = m(\{ x : |f_n| = n \}) = \frac{1}{2^{\left\lfloor \log_2 n \right\rfloor}} \to 0
\]
as \( n \to \infty \).

(b) Does \( f_n \to 0 \) in the \( L^1 \) norm?

No, since for all \( n \in \mathbb{N} \),
\[
\int_0^1 f_n = \frac{n}{2^{\left\lfloor \log_2 n \right\rfloor}} \geq 1.
\]

(c) Does \( f_n \to 0 \) pointwise \( m \)-almost everywhere?

No, since for each \( \epsilon > 0 \),
\[
m \left( \limsup_{n \to \infty} \{ x : |f_n| > \epsilon \} \right) \\
= m \left( \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \{ x : |f_n| > \epsilon \} \right) \\
= \lim_{k \to \infty} m \left( \bigcup_{n=k}^{\infty} \{ x : |f_n| > \epsilon \} \right) \\
= \lim_{k \to \infty} m \left( \bigcup_{q=0}^{2^p-1} \left[ \frac{q}{2^p}, \frac{q+1}{2^p} \right] \right) \\
= \lim_{k \to \infty} m([0, 1]) \\
= 1,
\]
where we have used downward monotone convergence in this finite-measure setting.
2. (i) Show that every function $F : \mathbb{R} \to \mathbb{R}$ of bounded variation is bounded and that
\[
\lim_{x \to \infty} F(x) \quad \text{and} \quad \lim_{x \to -\infty} F(x)
\]
exist. Let $a$ and $b$ be distinct real numbers. From the triangle inequality we know that
\[
|F(b)| \leq |F(a)| + |F(a) - F(b)| \leq |F(a)| + |F|_{TV(\mathbb{R})} < \infty.
\]
On holding $a$ fixed and varying $b$, we see that $F$ is bounded on $\mathbb{R}$. Furthermore, in view of Theorem 5.4, we can express $F$ as the difference of two bounded monotone functions. Since the limit of each bounded monotone function exists as $x \to \infty$ (or $x \to -\infty$), the same holds true for their difference.

(ii) Provide a counterexample of a bounded, continuous function $F : \mathbb{R} \to \mathbb{R}$ with bounded support that does not have bounded variation.

The bounded continuous function
\[
F(x) = \begin{cases} 
    x \sin \frac{1}{x} & \text{if } x \in \left[ -\frac{1}{\pi}, 0 \right) \cup (0, \frac{1}{\pi}], \\
    0 & \text{otherwise}
\end{cases}
\]
has infinite total variation since the sum
\[
\sum_{i=1}^{n} \left| F\left( \frac{2}{(2i+1)\pi} \right) - F\left( \frac{2}{(2i-1)\pi} \right) \right| = \sum_{i=1}^{n} \left[ \frac{2}{(2i+1)\pi} + \frac{2}{(2i-1)\pi} \right]
\]
diverges as $n \to \infty$ (being the sum of two positive divergent series).

3. Let $[a, b] \subset \mathbb{R}$ be compact, $D$ be a nonempty subset of $\mathbb{R}$, and $f : D \times [a, b] \to \mathbb{R}$ have the properties:

(i) for every $y \in [a, b]$ the function $D \to \mathbb{R}$, $x \mapsto f(x, y)$ is continuous;
(ii) for every $x \in D$, the function $h_x : [a, b] \to \mathbb{R}$, $y \mapsto f(x, y)$ is Lebesgue measurable;
(iii) there is an absolutely Lebesgue integrable function $g : [a, b] \to [0, \infty)$ such that $|f(x, y)| \leq g(y)$ for all $x \in D$ and $y \in [a, b]$.

Show that the function $F : D \to \mathbb{R}$, $x \mapsto \int_{a}^{b} f(x, y) \, dy$ is continuous.

Hint: consider a sequence $\{x_n\}_{n=1}^{\infty}$ in $D$.

Fix $x \in D$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $D$ with $\lim_{n \to \infty} x_n = x$. Then (i) implies that $h_x = \lim_{n \to \infty} f(x_n, y)$, while (iii) yields $|f(x_n, y)| \leq g(y)$ on $[a, b]$. From the dominated convergence theorem, we conclude that
\[
F(x) = \int_{a}^{b} h_x(y) \, dy = \int_{a}^{b} \lim_{n \to \infty} f(x_n, y) \, dy = \lim_{n \to \infty} \int_{a}^{b} f(x_n, y) \, dy = \lim_{n \to \infty} F(x_n).
\]
Thus $F$ is continuous at each $x \in D$.  

3
4. Consider the measure spaces \(X = Y = (\mathbb{N}, \mathcal{P}(\mathbb{N}), \#)\), where \(\#\) is the counting measure and define the function \(f : X \times Y \to \mathbb{R}\) by

\[
f(i, j) = \begin{cases} 
1 & \text{if } j = i, \\
-1 & \text{if } j = i + 1, \\
0 & \text{otherwise.}
\end{cases}
\]

(a) Evaluate 
\[
\int_X \int_Y f(i, j) \, d\#(j) \, d\#(i)
\]

\[
= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f(i, j) = \sum_{i=1}^{\infty} [f(i, i) + f(i, i + 1)] = \sum_{i=1}^{\infty} [1 - 1] = 0.
\]

(b) Evaluate 
\[
\int_Y \int_X f(i, j) \, d\#(i) \, d\#(j)
\]

\[
= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} f(i, j) = f(1, 1) + \sum_{j=2}^{\infty} [f(j, j) + f(j - 1, j)] = 1 + \sum_{j=2}^{\infty} [1 - 1] = 1.
\]

(c) Are your results explained by Tonelli’s theorem? Why or why not?
No, Tonelli’s theorem does not apply since \(f\) is not unsigned.

(d) Are your results explained by Fubini’s theorem? Why or why not?
No, Fubini’s theorem does not apply since \(f\) is not absolutely integrable with respect to \(\# \times \#: \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |f(i, j)| = \infty\).