1. (Cantor Function) Define the functions $F_0, F_1, F_2, \ldots : [0, 1] \to \mathbb{R}$ recursively: let $F_0(x) = x$ for $x \in [0, 1]$ and for $n \in \mathbb{N}$ define

$$F_n(x) = \begin{cases} \frac{1}{2} F_{n-1}(3x) & \text{if } x \in [0, \frac{1}{3}]; \\ \frac{1}{2} & \text{if } x \in (\frac{1}{3}, \frac{2}{3}); \\ \frac{1}{2} + \frac{1}{2} F_{n-1}(3x - 2) & \text{if } x \in [\frac{2}{3}, 1]. \end{cases}$$

(i) Graph $F_0, F_1, F_2,$ and $F_3$ on a single graph.

(ii) Using induction, show for each $n = 0, 1, \ldots$ that $F_n$ is a continuous monotone increasing function with $F_n(0) = 0$ and $F_n(1) = 1$.

We note that $F_0(x) = x$ is continuous and increasing, with $F_0(0) = 0$ and $F_0(1) = 1$. For $n \in \mathbb{N}$, assume that $F_{n-1}(x)$ is continuous and increasing, with $F_{n-1}(0) = 0$ and $F_{n-1}(1) = 1$. Since the composition of continuous increasing functions is also continuous and increasing, $F_n$ is continuous and increasing on all three intervals $[0, 1/3]$, $(1/3, 2/3)$, and $[2/3, 1]$. Moreover, $F_n(0) = \frac{1}{2} F_{n-1}(0) = 0$ and $F_n(1) = \frac{1}{2} + \frac{1}{2} F_{n-1}(1) = 1$. The desired result then follows by induction.

(iii) Show for each $n = 0, 1, \ldots$ and $x \in [0, 1]$ that $|F_{n+1}(x) - F_n(x)| \leq 2^{-n}$. Conclude that $\{F_n\}_{n=1}^\infty$ converges uniformly to a limit $F : [0; 1] \to \mathbb{R}$. The limit $F(x)$, known as the Cantor function, expresses the fraction of the “mass” of the Cantor set in $[0, x]$.
We note that

\[
|F_1(x) - F_0(x)| = \begin{cases} 
\frac{1}{2}|x| & \text{if } x \in [0, \frac{1}{3}]; \\
\frac{1}{2} - x & \text{if } x \in \left(\frac{1}{3}, \frac{2}{3}\right); \\
\frac{1}{2}|x - 1| & \text{if } x \in \left[\frac{2}{3}, 1\right]. 
\end{cases}
\]

(1)

Assume that \(|F_n(x) - F_{n-1}(x)| \leq 2^{-n}\) for some \(n \in \mathbb{N} \cup \{0\}\). Then

\[
|F_{n+1}(x) - F_n(x)| = \begin{cases} 
\frac{1}{2}|F_n(3x) - F_{n-1}(3x)| & \text{if } x \in [0, \frac{1}{3}]; \\
0 & \text{if } x \in \left(\frac{1}{3}, \frac{2}{3}\right); \\
\frac{1}{2}|F_n(3x - 2) - F_{n-1}(3x - 2)| & \text{if } x \in \left[\frac{2}{3}, 1\right], 
\end{cases}
\]

so that \(|F_{n+1}(x) - F_n(x)| \leq 2^{-(n+1)}\). By induction, we conclude that \(|F_n(x) - F_{n-1}(x)| \leq 2^{-n}\) for all \(n \in \mathbb{N}\).

Observe for \(n > m\) that

\[
|F_n(x) - F_m(x)| = \left| \sum_{k=m+1}^{n} [F_k(x) - F_{k-1}(x)] \right| 
\leq \sum_{k=m+1}^{n} |F_k(x) - F_{k-1}(x)| 
\leq \sum_{k=m+1}^{n} 2^{-k} \leq 2^{-m} \sum_{k=1}^{\infty} 2^{-k} = 2^{-m}
\]

uniformly on \([0, 1]\) as \(m \to \infty\). We thus see on \([0, 1]\) that \(\{F_n\}_{n=1}^{\infty}\) is a uniform Cauchy sequence that converges uniformly to some limit \(F : [0, 1] \to \mathbb{R}\).

(iv) Show that the Cantor function \(F\) is continuous and monotone increasing, with \(F(0) = 0\) and \(F(1) = 1\).

Since \(F\) is the uniform limit of a sequence of continuous functions, it is also continuous:

\[
|F(x) - F(a)| \leq |F(x) - F_n(x)| + |F_n(x) - F_n(a)| + |F_n(a) - F(a)|.
\]

To show that \(F\) is increasing, fix \(x < y\) and consider

\[
F(y) - F(x) = \lim_{n \to \infty} F_n(y) - F_n(x) \geq 0.
\]

Finally \(F(0) = \lim_{n \to \infty} F_n(0) = 0\) and \(F(1) = \lim_{n \to \infty} F_n(1) = 1\).
(v) Show that if \( x \in [0, 1] \) lies outside the Cantor set \( C \), then \( F \) is constant in a neighbourhood of \( x \), so that \( F'(x) = 0 \). Conclude that \( \int_{[0,1]} F'(x) \, dx = 0 \neq 1 = F(1) - F(0) \) and hence the fundamental theorem of calculus fails.

For \( n \in \mathbb{N} \) consider the auxiliary function

\[
F_n(x) = \begin{cases} 
F_{n-1}(3x) & \text{if } x \in \left[0, \frac{1}{3}\right]; \\
\frac{1}{2} & \text{if } x \in \left(\frac{1}{3}, \frac{2}{3}\right); \\
F_{n-1}(3x - 2) & \text{if } x \in \left[\frac{2}{3}, 1\right],
\end{cases}
\]

with \( F_0(x) = x \). If \( x \in [0, 1] \setminus C \), then it has at least one 1 in its ternary representation, where we adopt the convention that we always write ternary expansions so as to avoid using the digit 1 where possible (e.g. 1/3 is represented as 0.02 rather than 0.1); this corresponds to the convention in constructing the Cantor set that only the interior of each middle third is removed. Let \( n \in \mathbb{N} \) be the position (after the ternary point) of the first such digit 1. Since each map \( x \to 3x \) shifts the ternary digits in the representation of \( x \) by one place to the left and the map \( x \to x - 2 \) removes each leading digit 2, we see that \( F_n(x) = 1/2 \). Moreover, since each \( x \in [0, 1] \setminus C \) lies in the interior of some middle-third subinterval, \( F_n(x) = 1/2 \) throughout a neighbourhood of \( x \). We thus see that \( F_n \), and hence the original function \( F \), is locally constant at each \( x \in [0, 1] \setminus C \) and therefore differentiable at \( x \), with derivative 0. Since \( m(C) = 0 \), we see that \( F \) is differentiable almost everywhere, with \( \int_{[0,1]} F'(x) \, dx = 0 \) even though \( F(1) - F(0) = 1 - 0 = 0 \).

(vi) Show that \( F\left(\sum_{k=1}^{\infty} a_k 3^{-k}\right) = \sum_{k=1}^{\infty} \frac{a_k}{2} 2^{-k} \) for any digits \( a_1, a_2, \ldots \in \{0, 2\} \).

This follows from the ternary representation of \( x \in C \) (which consists of only the ternary digits 0 and 2), noting that whenever \( a_k = 2 \) (so that \( a_k/2 = 1 \)), we get a contribution of \( 2^{-k} \) from the third case in the definition of \( F_n \).

(vii) Let \( I_n = [\sum_{k=1}^{n} a_k 3^{-k}, 3^{-n} + \sum_{k=1}^{n} a_k 3^{-k}] \) for \( n \geq 0 \) and \( a_1, \ldots, a_n \in \{0, 2\} \). Show that \( I_n \) is an interval of length \( 3^{-n} \), but \( F(I_n) \) is an interval of length \( 2^{-n} \).

Motivated by the fact that \( 3^{-n} = 2 \sum_{k=n+1}^{\infty} 3^{-k} \), we define \( a_k = 2 \) for \( k \geq n+1 \), so that \( I_n = [\sum_{k=1}^{n} a_k 3^{-k}, \sum_{k=1}^{\infty} a_k 3^{-k}] \). Since \( F \) is monotonic we know from part (vi) that \( F(I_n) \) is an interval of length

\[
F\left(\sum_{k=1}^{\infty} a_k 3^{-k}\right) - F\left(\sum_{k=1}^{n} a_k 3^{-k}\right) = \sum_{k=1}^{\infty} \frac{a_k}{2} 2^{-k} - \sum_{k=1}^{n} \frac{a_k}{2} 2^{-k} = \sum_{k=n+1}^{\infty} 2^{-k} = 2^{-n}.
\]
(viii) Show that $F$ is not differentiable at any element of the Cantor set $C$.

Let $x \in C$. Then $x$ belongs to every interval $I_n = [c_n, d_n]$ in part (vii). If $F$ were differentiable at $x$, then given $\epsilon > 0$ there would exist $N \in \mathbb{N}$ such that

$$|F(d_n) - F(x) - F'(x)(d_n - x)| \leq \epsilon (d_n - x)$$

and

$$|F(x) - F(c_n) - F'(x)(x - c_n)| \leq \epsilon (x - c_n)$$

whenever $n > N$, so that

$$|F(d_n) - F(c_n) - F'(x)(d_n - c_n)| \leq |F(d_n) - F(x) - F'(x)(d_n - x)|$$

$$+ |F(x) - F(c_n) - F'(x)(x - c_n)| \leq \epsilon (d_n - c_n).$$

Part (vii) would then imply that

$$|2^{-n} - F'(x)3^{-n}| \leq \epsilon 3^{-n},$$

or equivalently,

$$|(3/2)^n - F'(x)| \leq \epsilon,$$

whenever $n > N$, which would contradict the fact that $\lim_{n \to \infty} (3/2)^n = \infty$. Hence $F$ is not differentiable at any $x \in C$.

2. Prove or provide a counterexample (and justify): if $F$ is a monotonic function on $[0, 1]$ with almost-everywhere defined derivative $f$, then

$$\int_0^1 f(x) \, dx = F(1) - F(0).$$

False: let $F$ be the Cantor function, which is constant almost everywhere (except on the Cantor set) and hence $\int_0^1 f(x) \, dx = 0 \neq 1 = F(1) - F(0)$.

Alternatively, consider $1_{[0,1]}$.

3. (i) Show that every function $F : \mathbb{R} \to \mathbb{R}$ of bounded variation is bounded and that $\lim_{x \to \infty} F(x)$ and $\lim_{x \to -\infty} F(x)$ exist.

Let $a$ and $b$ be distinct real numbers. From the triangle inequality we know that

$$|F(b)| \leq |F(a)| + |F(b) - F(a)| \leq |F(a)| + |F|_{TV(\mathbb{R})} < \infty.$$ 

On holding $a$ fixed and varying $b$, we see that $F$ is bounded on $\mathbb{R}$.

Furthermore, in view of Theorem 5.4, we can express $F$ as the difference of two bounded monotone functions. Since the limit of each bounded monotone function exists as $x \to \infty$ (or $x \to -\infty$), the same holds true for their difference.
(ii) Provide a counterexample of a bounded, continuous function $F : \mathbb{R} \to \mathbb{R}$ with bounded support that does not have bounded variation.

The bounded continuous function

$$F(x) = \begin{cases} 
  x \sin \frac{1}{\pi} & \text{if } x \in [-\frac{1}{\pi}, 0) \cup (0, \frac{1}{\pi}], \\
  0 & \text{otherwise}
\end{cases}$$

has infinite total variation since the sum

$$\sum_{i=1}^{n} \left| F\left(\frac{2}{(2i+1)\pi}\right) - F\left(\frac{2}{(2i-1)\pi}\right) \right| = \sum_{i=1}^{n} \left[ \frac{2}{(2i+1)\pi} + \frac{2}{(2i-1)\pi} \right]$$

diverges as $n \to \infty$ (being the sum of two positive divergent series).

4. Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $\alpha : [a, b] \to \mathbb{R}$ be a function of bounded variation. Show that the function

$$F_\alpha : [a, b] \to \mathbb{R}, \quad x \mapsto \int_{a}^{x} f(t) \, d\alpha(t)$$

is also of bounded variation. (Hint: Consider first the case where $\alpha$ is increasing.)

Since $f$ is continuous on $[a, b]$, there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$. Suppose first that $\alpha$ is increasing. On the partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$,

$$\sum_{i=1}^{n} |F_\alpha(x_i) - F_\alpha(x_{i-1})| = \sum_{i=1}^{n} \left| \int_{a}^{x_i} f(t) \, d\alpha(t) - \int_{a}^{x_{i-1}} f(t) \, d\alpha(t) \right|$$

$$= \sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} f(t) \, d\alpha'(t) \right|$$

$$\leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(t)|\alpha'(t) \, dt \quad \text{since } \alpha \text{ is increasing},$$

$$= \int_{a}^{b} |f(t)|\alpha'(t) \, dt \leq M \int_{a}^{b} \alpha'(t) \, dt \leq M[\alpha(b) - \alpha(a)],$$

on making use of Theorem 5.6. It follows that $|F_\alpha|_{TV([a, b])} < \infty$; that is, $F_\alpha$ is of bounded variation.

Let $\alpha$ now be arbitrary. Then there exists increasing functions $\beta, \gamma : [a, b] \to \mathbb{R}$ such that $\alpha = \beta - \gamma$. Since $F_\beta$ and $F_\gamma$ are of bounded variation by the foregoing, the same is true of $F_\alpha = F_\beta - F_\gamma$. 


5. Consider the measure spaces $X = Y = (\mathbb{N}, \mathcal{P}(\mathbb{N}), \#)$, where $\#$ is the counting measure and define the function $f : X \times Y \to \mathbb{R}$ by

$$f(i, j) = \begin{cases} 1 & \text{if } j = i, \\ -1 & \text{if } j = i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Evaluate $\int_X \int_Y f(i, j) \, d\#(j) \, d\#(i)$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f(i, j) = \sum_{i=1}^{\infty} [f(i, i) + f(i, i + 1)] = \sum_{i=1}^{\infty} [1 - 1] = 0.$$  

(b) Evaluate $\int_Y \int_X f(i, j) \, d\#(i) \, d\#(j)$

$$= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} f(i, j) = f(1, 1) + \sum_{j=2}^{\infty} [f(j, j) + f(j - 1, j)] = 1 + \sum_{j=2}^{\infty} [1 - 1] = 1.$$ 

(c) Are your results explained by Tonelli’s theorem? Why or why not?

No, Tonelli’s theorem does not apply since $f$ is not unsigned.

(d) Are your results explained by Fubini’s theorem? Why or why not?

No, Fubini’s theorem does not apply since $f$ is not absolutely integrable with respect to $\# \times \#: \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |f(i, j)| = \infty.$