1. Let \((X, \mathcal{B}, \mu)\) be a measure space and for each \(n \in \mathbb{N}\) let \(f_n : X \to \mathbb{C}\) and \(f : X \to \mathbb{C}\) be measurable functions.

   (i) If \(f_n\) converges to \(f\) uniformly, then \(f_n\) converges to \(f\) pointwise.

   We are given that for every \(\epsilon > 0\) there exists \(N = N(\epsilon)\) such that \(|f_n(x) - f(x)| < \epsilon\) for all \(x \in X\) whenever \(n \geq N\). In particular, this implies that each \(x \in X\), \(f_n\) converges pointwise to \(f\).

   (ii) If \(f_n\) converges to \(f\) uniformly, then \(f_n\) converges to \(f\) in the \(L^\infty\) norm. Conversely, if \(f_n\) converges to \(f\) in the \(L^\infty\) norm, then \(f_n\) converges to \(f\) uniformly outside of a null set (i.e. there exists a null set \(E\) such that the restriction \(f_n|_{X \setminus E}\) of \(f_n\) to the complement of \(E\) converges to the restriction \(f|_{X \setminus E}\) of \(f\)).

   We are given that for every \(\epsilon > 0\) there exists \(N = N(\epsilon)\) such that \(|f_n(x) - f(x)| < \epsilon\) for all \((\text{and hence for almost all})\) \(x \in X\) whenever \(n \geq N\). Thus \(f_n\) converges to \(f\) in the \(L^\infty\) norm (uniformly almost everywhere).

   By definition, if \(f_n\) converges to \(f\) in the \(L^\infty\) norm, there is a subset \(E\) of \(X\) with \(\mu(E) = 0\) such that for every \(\epsilon > 0\), we have \(|f_n(x) - f(x)| < \epsilon\) whenever \(n\) is larger than some threshold \(N(\epsilon)\) and \(x \in X \setminus E\). That is, \(f_n\) converges uniformly to \(f\) outside of a null set \(E\).

   (iii) If \(f_n\) converges to \(f\) in the \(L^\infty\) norm, then \(f_n\) converges to \(f\) almost uniformly.

   Given \(\epsilon > 0\), we know from (ii) that \(f_n\) converges uniformly to \(f\) outside of a set \(E\) such that \(\mu(E) = 0 < \epsilon\).

   (iv) If \(f_n\) converges to \(f\) almost uniformly, then \(f_n\) converges to \(f\) pointwise almost everywhere.

   For each \(k \in \mathbb{N}\) there exists a set \(E_k\) such that \(\mu(E_k) < 1/k\) and \(f_n\) converges uniformly to \(f\) on \(X \setminus E_k\). Consider \(E = \bigcap_{k=1}^\infty E_k\). By monotonicity, \(\mu(E) \leq \mu(E_k) < 1/k\) for every \(k \in \mathbb{N}\). Hence \(\mu(E) = 0\). Let \(x \in X \setminus E\). Then \(x \in X \setminus E_k\) for some \(k \in \mathbb{N}\). Since \(f_n\) converges uniformly to \(f\) on \(X \setminus E_k\), it converges pointwise to \(f\) at \(x\). That is, \(f_n\) converges pointwise to \(f\) on \(X \setminus E\).

   (v) If \(f_n\) converges to \(f\) pointwise, then \(f_n\) converges to \(f\) pointwise almost everywhere.

   This follows from the fact that “everywhere” implies “almost everywhere.”

   (vi) If \(f_n\) converges to \(f\) in the \(L^1\) norm, then \(f_n\) converges to \(f\) in measure.

   From Markov’s inequality, we know for each \(\epsilon > 0\) that

   \[
   \mu(\{x \in \mathbb{R}^d : |f_n(x) - f(x)| \geq \epsilon\}) \leq \frac{1}{\epsilon} \int_X |f_n - f| \, d\mu \to 0
   \]
as \( n \to \infty \).

Alternative solution: Given \( \epsilon \geq 0 \). Let \( S_{\epsilon,n} = \{ x \in X : |f_n(x) - f(x)| \geq \epsilon \} \) for \( n \in \mathbb{N} \). If \( \lim_{n \to \infty} \mu(S_{\epsilon,n}) \neq 0 \), then for every \( \epsilon' > 0 \) there would exist a subsequence \( \{n_k\}_{k=1}^{\infty} \) such that \( \mu(S_{\epsilon,n_k}) \geq \epsilon' \) for every \( k \in \mathbb{N} \). Choose \( N \) such that \( k > N \Rightarrow \int_X |f - f_{n_k}| \, d\mu < \epsilon \epsilon' \). This would contradict \( \int_X |f - f_{n_k}| \, d\mu \geq \epsilon \mu(S_{\epsilon,n_k}) \geq \epsilon \epsilon' \).

Thus \( \lim_{n \to \infty} \mu(S_{\epsilon,n}) = 0 \).

(vii) If \( f_n \) converges to \( f \) almost uniformly, then \( f_n \) converges to \( f \) in measure.

Given \( \epsilon > 0 \), there exists \( N \in \mathbb{N} \) and an exceptional set \( E_\epsilon \) with measure less than \( \epsilon \) such that \( |f_n(x) - f(x)| < \epsilon \) whenever \( n > N \) for all \( x \in X \setminus E_\epsilon \). Then \( \{ x \in X : |f_n(x) - f(x)| \geq \epsilon \} \subset E_\epsilon \) has measure less than \( \epsilon \).

2. (i) Show that every function \( F : \mathbb{R} \to \mathbb{R} \) of bounded variation is bounded and that \( \lim_{x \to \infty} F(x) \) and \( \lim_{x \to -\infty} F(x) \) exist.

Let \( a \) and \( b \) be distinct real numbers. From the triangle inequality we know that
\[
|F(b)| \leq |F(a)| + |F(b) - F(a)| \leq |F|_{TV(\mathbb{R})} < \infty.
\]

On holding \( a \) fixed and varying \( b \), we see that \( F \) is bounded on \( \mathbb{R} \).

Furthermore, in view of Theorem 1.19, we can express \( F \) as the difference of two bounded monotone functions. Since the limit of each bounded monotone function exists as \( x \to \infty \) (or \( x \to -\infty \)), the same holds true for their difference.

(ii) Provide a counterexample of a bounded, continuous function \( F : \mathbb{R} \to \mathbb{R} \) with bounded support that does not have bounded variation.

The bounded continuous function
\[
F(x) = \begin{cases} 
  x \sin \frac{1}{x} & \text{if } x \in [-\frac{1}{\pi},0) \cup (0, \frac{1}{\pi}], \\
  0 & \text{otherwise}
\end{cases}
\]

has infinite total variation since the sum
\[
\sum_{i=1}^{n} \left| F\left( \frac{2}{(2i+1)\pi} \right) - F\left( \frac{2}{(2i-1)\pi} \right) \right| = \sum_{i=1}^{n} \left[ \frac{2}{(2i+1)\pi} + \frac{2}{(2i-1)\pi} \right]
\]
diverges as \( n \to \infty \) (being the sum of two positive divergent series).

3. Show that every convex function \( f : \mathbb{R} \to \mathbb{R} \) is continuous and almost everywhere differentiable, with derivative almost everywhere equal to an increasing function.

Recall that the convexity condition can be re-expressed in terms of the slope of a secant:
\[
\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a} \leq \frac{f(b) - f(x)}{b - x} \quad \forall x \in (a,b), \quad \forall a \neq b \in \mathbb{R}.
\]
Applying this criterion repeatedly, we see for all real numbers $A < a < x < y < b < B$ that

$$\frac{f(a) - f(A)}{a - A} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(B) - f(b)}{B - b}.$$ 

On letting

$$C = \max \left( \left| \frac{f(a) - f(A)}{a - A} \right|, \left| \frac{f(B) - f(b)}{B - b} \right| \right),$$

we thus see that $f$ is locally Lipschitz, and therefore continuous, on every compact interval $[a, b] \subset \mathbb{R}$:

$$|f(x) - f(y)| \leq C|x - y| \quad \forall x, y \in [a, b].$$

Moreover, $f$ is locally of bounded variation and hence differentiable almost everywhere.

Let

$$m(x) = \frac{f(x) - f(a)}{x - a} \quad (x \neq a), \quad M(x) = \frac{f(b) - f(x)}{b - x} \quad (x \neq b).$$

From convexity, we know that

$$m(x) \leq m(b) = M(a) \leq M(x)$$

whenever $a < x < b$. At points $a$ and $b$ where $f$ is differentiable we then see that

$$f'(a) = \lim_{x \to a} m(x) \leq m(b) = M(a) \leq \lim_{x \to b} M(x) = f'(b).$$

4. (Cantor Function) Define the functions $F_0, F_1, F_2, \ldots : [0, 1] \to \mathbb{R}$ recursively: let $F_0(x) \equiv x$ for $x \in [0, 1]$ and for $n \in \mathbb{N}$ define

$$F_n(x) \equiv \begin{cases} 
\frac{1}{2} F_{n-1}(3x) & \text{if } x \in [0, \frac{1}{3}]; \\
\frac{1}{2} & \text{if } x \in (\frac{1}{3}, \frac{2}{3}); \\
\frac{1}{2} + \frac{1}{2} F_{n-1}(3x - 2) & \text{if } x \in [\frac{2}{3}, 1].
\end{cases}$$

(i) Graph $F_0, F_1, F_2,$ and $F_3$ on a single graph.
(ii) Using induction, show for each \( n = 0, 1, \ldots \) that \( F_n \) is a continuous monotone increasing function with \( F_n(0) = 0 \) and \( F_n(1) = 1 \).

We note that \( F_0(x) = x \) is continuous and increasing, with \( F_0(0) = 0 \) and \( F_0(1) = 1 \). For \( n \in \mathbb{N} \), assume that \( F_{n-1}(x) \) is continuous and increasing, with \( F_{n-1}(0) = 0 \) and \( F_{n-1}(1) = 1 \). Since the composition of continuous increasing functions is also continuous and increasing, \( F_n \) is continuous and increasing on all three intervals \([0, 1/3], (1/3, 2/3), \text{ and } [1/3, 1]\). Moreover, \( F_n(0) = \frac{1}{2}F_{n-1}(0) = 0 \) and \( F_n(1) = \frac{1}{2} + \frac{1}{2}F_{n-1}(1) = 1 \). The desired result then follows by induction.

(iii) Show for each \( n = 0, 1, \ldots \) and \( x \in [0, 1] \) that \( |F_{n+1}(x) - F_n(x)| \leq 2^{-n} \). Conclude that \( \{F_n\}_{n=1}^{\infty} \) converges uniformly to a limit \( F : [0; 1] \to \mathbb{R} \). The limit \( F(x) \), known as the Cantor function, expresses the fraction of the “mass” of the Cantor set in \([0, x]\).

We note that

\[
|F_1(x) - F_0(x)| = \begin{cases} 
\frac{1}{2}|x| & \text{ if } x \in [0, \frac{1}{3}] ; \\
\frac{1}{2} - x & \text{ if } x \in (\frac{1}{3}, \frac{2}{3}) ; \\
\frac{1}{2}|x - 1| & \text{ if } x \in [\frac{2}{3}, 1] .
\end{cases}
\]

(1)

Assume that \( |F_n(x) - F_{n-1}(x)| \leq 2^{-n} \) for some \( n \in \mathbb{N} \cup \{0\} \). Then

\[
|F_{n+1}(x) - F_n(x)| = \begin{cases} 
\frac{1}{2}|F_n(3x) - F_{n-1}(3x)| & \text{ if } x \in [0, \frac{1}{3}] ; \\
0 & \text{ if } x \in (\frac{1}{3}, \frac{2}{3}) ; \\
\frac{1}{2}|F_n(3x - 2) - F_{n-1}(3x - 2)| & \text{ if } x \in [\frac{2}{3}, 1] ,
\end{cases}
\]

so that \( |F_{n+1}(x) - F_n(x)| \leq 2^{-(n+1)} \). By induction, we conclude that \( |F_n(x) - F_{n-1}(x)| \leq 2^{-n} \) for all \( n \in \mathbb{N} \).

Observe for \( n > m \) that

\[
|F_n(x) - F_m(x)| = \sum_{k=m+1}^{n} |F_k(x) - F_{k-1}(x)| 
\leq \sum_{k=m+1}^{n} |F_k(x) - F_{k-1}(x)| 
\leq \sum_{k=m+1}^{n} 2^{-k} < 2^{-m} \sum_{k=1}^{\infty} 2^{-k} = 2^{-m} \sum_{k=1}^{\infty} 2^{-k} = 2^{-m}
\]

uniformly on \([0, 1]\) as \( m \to \infty \). We thus see on \([0, 1]\) that \( \{F_n\}_{n=1}^{\infty} \) is a uniform Cauchy sequence that converges uniformly to some limit \( F : [0, 1] \to \mathbb{R} \).
(iv) Show that the Cantor function $F$ is continuous and monotone increasing, with $F(0) = 0$ and $F(1) = 1$.

Since $F$ is the uniform limit of a sequence of continuous functions, it is also continuous:

$$|F(x) - F(a)| \leq |F(x) - F_n(x)| + |F_n(x) - F_n(a)| + |F_n(a) - F(a)|.$$  

To show that $F$ is increasing, fix $x < y$ and consider

$$F(y) - F(x) = \lim_{n \to \infty} F_n(y) - F_n(x) \geq 0.$$  

Finally $F(0) = \lim_{n \to \infty} F_n(0) = 0$ and $F(1) = \lim_{n \to \infty} F_n(1) = 1$.

(v) Show that if $x \in [0, 1]$ lies outside the Cantor set $C$, then $F$ is constant in a neighbourhood of $x$, so that $F'(x) = 0$. Conclude that $\int_{[0,1]} F'(x)dx = 0 \neq 1 = F(1) - F(0)$ and hence the fundamental theorem of calculus fails. If $x \in [0,1] \setminus C$, then its has at least one 1 in its ternary representation, where we adopt the convention that we always write ternary expansions so as to avoid using the digit 1 where possible (e.g. $1/3$ is represented as $0.0\overline{2}$ rather than $0.1$); this corresponds to the convention in constructing the Cantor set that only the interior of each middle third is removed. Let $n \in \mathbb{N}$ be the position (after the ternary point) of the first such digit 1. Since each map $x \to 3x$ shifts the ternary digits in the representation of $x$ by one place to the left and the map $x \to x - 2$ removes each leading digit 2, we see that $F_n(x) = 1/2$. Moreover, since each $x \in [0,1] \setminus C$ lies in the interior of some middle-third subinterval, $F_n(x) = 1/2$ throughout a neighbourhood of $x$. Thus $F_n$ is locally constant at each $x \in [0,1] \setminus C$ and therefore differentiable at $x$, with derivative 0. Since $m(C) = 0$, we see that $F$ is differentiable almost everywhere, with $\int_{[0,1]} F'(x)dx = 0$ even though $F(1) - F(0) = 1 - 0 = 0$.

(vi) Show that $F \left( \sum_{k=1}^{\infty} a_k 3^{-k} \right) = \sum_{k=1}^{\infty} \frac{a_k}{2} 2^{-k}$ for any digits $a_1, a_2, \ldots \in \{0, 2\}$.

This follows from the ternary representation of $x \in C$ (which consists of only the ternary digits 0 and 2), noting that whenever $a_k = 2$ (so that $a_k/2 = 1$), we get a contribution of $2^{-k}$ from the third case in the definition of $F_n$.

(vii) Let $I_n = \left[ \sum_{k=1}^{n} a_k 3^{-k}, 3^{-n} + \sum_{k=1}^{n} a_k 3^{-k} \right]$ for $n \geq 0$ and $a_1, \ldots, a_n \in \{0, 2\}$. Show that $I_n$ is an interval of length $3^{-n}$, but $F(I_n)$ is an interval of length $2^{-n}$.

Motivated by the fact that $3^{-n} = 2 \sum_{k=n+1}^{\infty} 3^{-k}$, we define $a_k = 2$ for $k \geq n+1$, so that $I_n = \left[ \sum_{k=1}^{n} a_k 3^{-k}, \sum_{k=1}^{\infty} a_k 3^{-k} \right]$ Since $F$ is monotonic we know from part (vi) that $F(I_n)$ is an interval of length

$$F \left( \sum_{k=1}^{\infty} a_k 3^{-k} \right) - F \left( \sum_{k=1}^{n} a_k 3^{-k} \right) = \sum_{k=1}^{\infty} \frac{a_k}{2} 2^{-k} - \sum_{k=1}^{n} \frac{a_k}{2} 2^{-k} = \sum_{k=n+1}^{\infty} 2^{-k} = 2^{-n}.$$
(viii) Show that $F$ is not differentiable at any element of the Cantor set $C$.

Let $x \in C$. Then $x$ belongs to every interval $I_n = [c_n, d_n]$ in part (vii). If $F$ were differentiable at $x$, then given $\epsilon > 0$ there would exist $N \in \mathbb{N}$ such that

$$|F(d_n) - F(x) - F'(x)(d_n - x)| < \epsilon(d_n - x)$$

and

$$|F(x) - F(c_n) - F'(x)(x - c_n)| < \epsilon(x - c_n)$$

whenever $n > N$, so that

$$|F(d_n) - F(c_n) - F'(x)(d_n - c_n)| \leq |F(d_n) - F(x) - F'(x)(d_n - x)| + |F(x) - F(c_n) - F'(x)(x - c_n)| < \epsilon(d_n - c_n).$$

Part (vii) would then imply that

$$|2^{-n} - F'(x)3^{-n}| < \epsilon3^{-n},$$

or equivalently,

$$|(3/2)^n - F'(x)| < \epsilon,$$

whenever $n > N$, which would contradict the fact that $\lim_{n \to \infty} (3/2)^n = \infty$. Hence $F$ is not differentiable at any $x \in C$. 