1. Let $S$ be a Lebesgue measurable subset of $\mathbb{R}^d$. Use the fact that
\[ m^*(E \cap S) + m^*(E \cap S^c) = m(E) \]
for every elementary set $E$ to show the stronger result that
\[ m^*(A \cap S) + m^*(A \cap S^c) = m^*(A) \]
for every arbitrary subset $A$ of $\mathbb{R}^d$.

Hint: Begin with the definition of $m^*(A)$. Use subadditivity and monotonicity. Consider the cases where $m^*(A) < \infty$ and $m^*(A) = \infty$.

If $m^*(A)$ is finite, given $\epsilon > 0$, there exists a countable family of disjoint boxes $B_1, B_2, \ldots$ that cover $A$ such that
\[ \sum_{k=1}^{\infty} |B_k| < m^*(A) + \epsilon. \]

Using subadditivity and upward monotone convergence, we find
\begin{align*}
m^*(A) &\leq m^*(A \cap S) + m^*(A \cap S^c) \\
&\leq m\left(\bigcup_{k=1}^{\infty} B_k \cap S\right) + m\left(\bigcup_{k=1}^{\infty} B_k \cap S^c\right) \\
&= \lim_{n \to \infty} \left[ m\left(\bigcup_{k=1}^{n} B_k \cap S\right) + m\left(\bigcup_{k=1}^{n} B_k \cap S^c\right) \right] \\
&= \lim_{n \to \infty} m\left(\bigcup_{k=1}^{n} B_k\right) = m\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} |B_k| < m^*(A) + \epsilon.
\end{align*}

Since this holds for all $\epsilon > 0$, the desired result follows.

The result also holds when $m^*(A) = \infty$, since subadditivity then implies that
\[ \infty \leq m^*(A \cap S) + m^*(A \cap S^c) \leq \infty. \]

2. Let $X$ be an infinite set and let $\mathcal{F}$ be the collection of finite subsets of $X$. Show that $\mathcal{F}$ is not a $\sigma$-algebra.

Let $x \in X$. Then $\{x\} \subset \mathcal{F}$, but its complement in $X$ is not.
3. Show that an intersection $\cap_{\alpha \in I} B_\alpha$ of $\sigma$-algebras $B_\alpha$, where $I$ is an index set, is itself a $\sigma$-algebra.

Let $B = \cap_{\alpha \in I} B_\alpha$. Since $\emptyset \in B_\alpha$ for every $\alpha \in I$ we know that $\emptyset \in B$. If $S \in B$ then we know for every $\alpha \in I$ that $S \in B_\alpha$ and hence $S^c \in B_\alpha$; this implies that $S^c \in B$. Finally, if $S_1, S_2, \ldots \in B$ then we know for every $\alpha \in I$ that $S_1, S_2, \ldots \in B_\alpha$ and hence $\bigcup_{k=1}^{\infty} S_k \in B_\alpha$; this implies that $\bigcup_{k=1}^{\infty} S_k \in B$. Thus, $B$ is a $\sigma$-algebra.

4. Let $(X, B)$ be a measurable space. Show that an unsigned function $f$ is measurable $\iff$ for every rational $r$ the set $\{x \in X : f(x) > r\}$ is measurable.

$\Rightarrow$ This follows immediately from Prob 3.8.

$\Leftarrow$ This also follows from Prob 3.8 since

$$\{x \in X : f(x) > a\} = \bigcup_{r \in \mathbb{Q}, r > a} \{x \in X : f(x) > r\}.$$

5. Let $X$ be an uncountable set and define

$$B = \{S \subset X : S \text{ or } S^c \text{ is countable}\}$$

and

$$\mu(S) = \begin{cases} 1 & \text{if } S \text{ is uncountable;} \\ 0 & \text{if } S \text{ is countable.} \end{cases}$$

Show that $(X, B, \mu)$ is a measure space.

We need to show that $B$ is a $\sigma$-algebra and that $\mu$ is countably additive measure. Since $\emptyset$ is countable, we know that $\emptyset \in B$. If $S \in B$ then either $S$ is countable or $S^c$ is countable; this implies that $S^c \in B$. If $S_1, S_2, \ldots \in B$ then either all of them are countable (in which case $\bigcup_{k=1}^{\infty} S_k$ is countable and hence in $B$) or the complement of at least one of them is countable (in which case $(\bigcup_{k=1}^{\infty} S_k)^c = \bigcap_{k=1}^{\infty} S_k^c$ is countable and hence $\bigcup_{k=1}^{\infty} S_k$ is in $B$). Thus $B$ is a $\sigma$-algebra.

Next, we show $\mu$ is a countably additive measure. Since $\emptyset$ is countable, we have $\mu(\emptyset) = 0$. We will use the fact that two disjoint elements $A$ and $B$ of $B$ cannot both be uncountable as then both $A^c \supset B$ and $A$ would then be uncountable, contradicting $A \in B$. Thus, if $S_1, S_2, \ldots$ are disjoint elements of $B$, then either all of them are countable (so that $\bigcup_{k=1}^{\infty} S_k$ is countable and hence $\mu(\bigcup_{k=1}^{\infty} S_k) = 0$ or $\sum_{k=1}^{\infty} \mu(S_k)$) or exactly one of them is uncountable (in which case $\mu(\bigcup_{k=1}^{\infty} S_k) = 1$ or $\sum_{k=1}^{\infty} \mu(S_k)$).