Math 417: Honours Real Variables I  
Winter, 2019  Assignment 3  
February 16, due February 25

1. What is wrong in the following argument?

Express the open set \((0, 1)\) as a union of open intervals \(I_x\) centered on each real \(x \in (0, 1)\):

\[
\bigcup_{x \in (0, 1)} I_x = (0, 1).
\]

Each interval \(I_x\) contains a rational number, which we can use as a label for that interval. Since there are only countably many rational numbers, there are only countably many labels. Hence the above union actually contains only a countable number of intervals, each centered at a real number. There are therefore only a countable number of real numbers.

2. Let \(S \subset \mathbb{R}^d\) such that

\[
m^*(E \cap S) + m^*(E \cap S^c) = m(E)
\]

for every elementary set \(E\). Show that \(S\) is Lebesgue measurable.

3. Suppose \(S_n \subset \mathbb{R}^d, n = 1, 2, \ldots\) are Lebesgue measurable sets that converge pointwise to a set \(S\).

   (i) Show that \(S\) is Lebesgue measurable. Hint: use the fact that \(1_S(x) = \liminf_{n \to \infty} 1_{S_n}(x) = \limsup_{n \to \infty} 1_{S_n}(x)\) to write \(S\) in terms of countable unions and intersections of \(S_n\).

   (ii) (Dominated convergence theorem) Suppose that the \(S_n\) are all contained in another Lebesgue measurable set \(F\) of finite measure. Show that \(m(S_n)\) converges to \(m(S)\). Hint: use the upward and downward monotone convergence theorems.

   (iii) Give a counterexample to show that the dominated convergence theorem fails if the \(S_n\) are not contained in a set of finite measure, even if we assume that the \(m(S_n)\) are all uniformly bounded.

4. Show that an unsigned function \(f : \mathbb{R}^d \to [0, \infty]\) is a simple function iff it is measurable and takes on finitely many values.

5. Let \(d, d' \in \mathbb{N}\).

   (a) If \(A \subset \mathbb{R}^d\) and \(B \subset \mathbb{R}^{d'}\), show that

\[
m^{d+d'}(A \times B) \leq m^d(A) m^{d'}(B),
\]

where \(m^d\) denotes \(d\)-dimensional Lebesgue measure.
(b) Let $A \subset \mathbb{R}^d$ and $B \subset \mathbb{R}^{d'}$ be Lebesgue measurable sets (not necessarily of finite measure). Show that $A \times B$ is Lebesgue measurable, with

$$m^{d+d'}(A \times B) = m^d(A)m^{d'}(B).$$

6. If $f : \mathbb{R}^d \to [0, \infty]$ is Lebesgue measurable, show that the Lebesgue measure of $
\{(x, y) \in \mathbb{R}^d \times \mathbb{R} : 0 \leq y \leq f(x)\}$ exists and equals $\int_{\mathbb{R}^d} f$. 