1. A set is *nowhere dense* if its closure has an empty interior. For example, the Cantor set \( C = \bigcap_{n=1}^{\infty} C_n \) is nowhere dense, where \( C_0 = [0,1] \) and 
\[
C_n = \frac{1}{3} C_{n-1} \cup \left( \frac{2}{3} + \frac{C_{n-1}}{3} \right)
\]
for \( n \in \mathbb{N} \).

(a) Show that \( C \) has Lebesgue measure zero.

Consider ternary representations that avoid using the digit 1 where possible (e.g. \( 1/3 \) is represented as \( 0.0\overline{2} \) rather than \( 0.1\overline{0} \)). Using induction, we observe that \( C_n \) is the set of numbers whose ternary representations have only 0 or 2 in the first \( n \) places. This implies that \( C_{n+1} \subset C_n \) and \( m(C_n) = 2^n/3^n \). On applying downward monotone convergence, noting that \( m(C_0) = 1 \) is finite, we see that \( m(C) = \lim_{n \to \infty} m(C_n) = 0 \).

*Alternative solution:* We claim that \( C_{n+1} \subset C_n \) and \( m(C_n) = 2^n/3^n \) for all \( n \in \mathbb{N} \). Since \( C_1 = [0,1/3] \cup [1/3,1] \subset C_0 \) and \( m(C_0) = 1 \) we see that this holds for \( n = 0 \). Assuming that it holds for all \( n = 0, \ldots, k-1 \), we know that \( \frac{1}{3} C_k \subset \frac{1}{3} C_{k-1} \) and hence
\[
C_{k+1} = \frac{1}{3} C_k \cup \left( \frac{2}{3} + \frac{C_k}{3} \right) \subset \frac{1}{3} C_{k-1} \cup \left( \frac{2}{3} + \frac{C_{k-1}}{3} \right) = C_k.
\]
Also, since \( C_{k-1}/3 \) is a subset of \([0,1/3]\), it is disjoint from \( 2/3 + C_{k-1}/3 \) and has measure \( 2^{k-1}/3^k \). Thus by finite additivity, \( m(C_k) = 2^k/3^k \). The claim then follow by induction. On applying downward monotone convergence, noting that \( m(C_0) = 1 \) is finite, we see that \( m(C) = \lim_{n \to \infty} m(C_n) = 0 \).

*Alternative solution:* We have seen that \( C \) can be constructed by removing from \([0,1]\) a countable union of intervals \( I_n \) with measure \( m([0,1] \setminus C) = \sum_{n=1}^{\infty} |I_n| = 1 \). Since this countable union of intervals is Lebesgue measurable, so is its complement \( C \) in \([0,1]\). By finite additivity \( m(C) + m([0,1] \setminus C) = m([0,1]) = 1 \). Thus \( m(C) = 0 \).

(b) For arbitrary \( \epsilon \in (0,1) \), construct a nowhere dense subset of \([0,1]\) with Lebesgue measure at least \( 1 - \epsilon \).

Hint: first try to remove a dense set from \([0,1]\) that has measure \( \leq 1/2 \).

Let \( \{q_1, q_2, \ldots\} \) be an enumeration of \( \mathbb{Q} \cap [0,1] \) and
\[
S = \bigcup_{k=1}^{\infty} \left[ q_k - 2^{-k-1} \epsilon, q_k + 2^{-k-1} \epsilon \right],
\]
which has measure at most
\[
2 \cdot 2^{-1} \sum_{k=1}^{\infty} 2^{-k} = \epsilon.
\]
Then from disjoint additivity we see that the set \([0, 1] \setminus S\) has measure at least \(1 - \epsilon\). This set is nowhere dense because its closure contains no rationals (and therefore has an empty interior).

**Alternative solution:** Consider the so-called fat Cantor (Smith–Volterra–Cantor) set \(C = \cap_{n=1}^{\infty} C_n\), where \(C_0 = [0, 1]\) and \(C_n = \epsilon C_{n-1} \cup (1 - \epsilon + \epsilon C_{n-1})\) for \(n \in \mathbb{N}\), where \(\epsilon = \min(\epsilon/2, 1/4)\). This set is nowhere dense and has measure at least \(1 - \epsilon/(1 - 2\epsilon) \geq 1 - 2\epsilon \geq 1 - \epsilon\).

2. Let \(S \subseteq \mathbb{R}\) be Lebesgue measurable. Prove that there exists a subset \(H\) of \(S\) such that \(m(H) = m(S)/2\). Hint: Consider the function \(f : [0, \infty) \to [0, \infty]: x \mapsto m(S \cap [-x, x])\).

   If \(m(S) = \infty\), we can choose \(H = S\): \(m(H) = \infty/2 = \infty\).

   Otherwise \(m(S)\) is finite and we can let \(f : [0, \infty) \to [0, \infty): x \mapsto m(S \cap [-x, x])\). For \(0 \leq x < y\), we know from monotonicity and disjoint additivity that

   \[
   f(x) \leq f(y) = m((S \cap [-x, x]) \cup (S \cap [-y, -x]) \cup (S \cap [x, y])) = m(S \cap [-x, x]) + m(S \cap [-y, -x]) + m(S \cap [x, y]) \leq f(x) + 2(y - x).
   \]

   Thus \(|f(y) - f(x)| \leq 2|y - x|\) for all \(x, y \in \mathbb{R}\); that is, \(f\) is continuous on \([0, \infty)\). Since \(f(0) = 0\) and \(\lim_{x \to \infty} f(x) = m(S)\), we can then apply the intermediate value theorem to find an \(x\) such that \(m(S \cap [-x, x]) = m(S)/2\).

3. Suppose \(S_n \subseteq \mathbb{R}^d\), \(n = 1, 2, \ldots\) are Lebesgue measurable sets that converge pointwise to a set \(S\).

   \(\text{(a)}\) Show that \(S\) is Lebesgue measurable. Hint: use the fact that \(1_S(x) = \liminf_{n \to \infty} 1_{S_n}(x) = \limsup_{n \to \infty} 1_{S_n}(x)\) to write \(S\) in terms of countable unions and intersections of \(S_n\).

   By definition, we are given that \(\liminf_{n \to \infty} 1_{S_n}(x) = \limsup_{n \to \infty} 1_{S_n}(x) = 1_S(x)\). This implies

   \[
   \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n = S.
   \]

   Being a countable intersection of a countable union of Lebesgue measurable sets, \(S\) is also a Lebesgue measurable set.

   \(\text{(b)}\) (Dominated convergence theorem) Suppose that the \(S_n\) are all contained in another Lebesgue measurable set \(F\) of finite measure. Show that \(m(S_n)\) converges to \(m(S)\). Hint: use the upward and downward monotone convergence theorems.
Consider the increasing sequence of sets \( U_k = \bigcap_{n=k}^{\infty} S_n \). By the upward monotone convergence theorem,

\[
m(S) = m\left(\bigcup_{k=1}^{\infty} U_k\right) = \lim_{n \to \infty} m(U_n).
\]

Now consider the decreasing sequence of sets \( T_k = \bigcup_{n=k}^{\infty} S_n \subset F \). By monotonicity, each of the sets \( T_k \) has finite measure. By the downward monotone convergence theorem,

\[
m(S) = m\left(\bigcap_{k=1}^{\infty} T_k\right) = \lim_{n \to \infty} m(T_n).
\]

For each \( n \in \mathbb{N} \), we also know that \( U_n \subset S_n \subset T_n \) and so by monotonicity, we have \( m(U_n) \leq m(S_n) \leq m(T_n) \). The desired result then follows from the squeeze principle.

(c) Give a counterexample to show that the dominated convergence theorem fails if the \( S_n \) are not contained in a set of finite measure, even if we assume that the \( m(S_n) \) are all uniformly bounded.

Consider \( S_n = [n, n+1] \) for \( n \in \mathbb{N} \). Note that \( S_n \) converge pointwise to \( \emptyset \), yet

\[
\lim_{n \to \infty} m(S_n) = \lim_{n \to \infty} 1 = 1 \neq 0 = m(\emptyset).
\]

4. Let \( f : \mathbb{R}^d \to \mathbb{C} \). Show that

(a) if \( f \) is continuous, it is measurable;

If \( f \) is continuous, the preimage \( f^{-1}(V) \) of every open set \( V \subset \mathbb{C} \) is itself open.

We then deduce from Lemma 2.3 (iv) that \( f \) is measurable.

(b) if \( f \) is almost everywhere equal to a measurable function, it is itself measurable

If \( f \) is almost everywhere equal to a measurable function, at almost every point, it is the pointwise limit of a sequence of unsigned simple functions. Lemma 2.3 (ii) then implies that \( f \) is measurable.

(c) if a sequence \( f_n \) of complex-valued measurable functions converges pointwise almost everywhere to \( f \), then \( f \) is measurable.

Given \( \epsilon > 0 \), for almost all \( x \in \mathbb{R}^d \) there exists \( N \in \mathbb{N} \) such that

\[
|f_n(x) - f(x)| < \epsilon
\]

whenever \( n > N \). Being measurable, each \( f_n \) is the pointwise limit of a sequence of simple functions \( \{f_{n,m}\}_{m=1}^{\infty} \); that is, there exists \( M_n \in \mathbb{N} \) such that

\[
|f_{n,m}(x) - f_n(x)| < \epsilon
\]

3
whenever $m > M_n$. Choose $m_n = M_n + 1$. From the triangle inequality, we then see that

$$|f_{n,m_n}(x) - f(x)| \leq |f_{n,m_n}(x) - f_n(x)| + |f_n(x) - f(x)| < 2\epsilon$$

whenever $n > N$. Thus, $\lim_{n \to \infty} f_{n,m_n}(x) = f(x)$ for almost all $x \in \mathbb{R}^d$ and hence by Lemma 2.3 (ii), $f$ is measurable.

(d) if $f$ is measurable, the composition $\phi \circ f$ of a continuous function $\phi : \mathbb{C} \to \mathbb{C}$ and $f$ is measurable.

Let $V$ be an open subset of $\mathbb{C}$. Since $\phi$ is continuous, function, the preimage $\phi^{-1}(V)$ is also open. Since $f$ is measurable, we see from Lemma 2.3 (iv) that $f^{-1}(\phi^{-1}(V)) = (\phi \circ f)^{-1}(V)$ is open. Using Lemma 2.3 (iv) once more, we thus see that $\phi \circ f$ is measurable.
5. If \( f : \mathbb{R}^d \to [0, \infty] \) is Lebesgue measurable, show that the Lebesgue measure of \( \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : 0 \leq y \leq f(x)\} \) exists and equals \( \int_{\mathbb{R}^d} f \).

Let \( G = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y = f(x)\} \). We claim that \( m(G) = 0 \) in \( \mathbb{R}^{d+1} \).

To see this, let \( B_k \) be a box of side \( k \in \mathbb{N} \) centered on the origin and let \( f_k \) be the restriction of \( f \) to \( B_k \). Let \( j, n \in \mathbb{N} \). The sets \( I_i = [j + \frac{j}{n}, j + \frac{j+1}{n}) \), \( i = 0, \ldots, n - 1 \) form a pairwise disjoint cover of \( [j, j + 1) \). By Lemma 2.2 (ix), the inverse images \( f_k^{-1}([j, j + 1]) \) and \( f_k^{-1}(I_i) \) are Lebesgue measurable. We then use subadditivity and Problem 1.21 (a) and (b):

\[
m^*(G \cap (B_k \times [j, j + 1))) \leq \sum_{i=0}^{n-1} m^*(G \cap (B_k \times I_i))
\leq \sum_{i=0}^{n-1} m(f_k^{-1}(I_i) \times I_i) = \frac{1}{n} \sum_{i=0}^{n-1} m(f_k^{-1}(I_i))
= \frac{1}{n} m(f_k^{-1}([j, j + 1])) \leq \frac{1}{n} m(B_k) \to 0
\]

as \( n \to \infty \) and hence

\[
m^*(G) \leq \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} m^*(G \cap (B_k \times [j, j + 1))) = 0.
\]

Thus \( m(G) = 0 \), as claimed, and it therefore suffices to show that \( m(S) = \int_{\mathbb{R}^d} f \), where

\[
S = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : 0 \leq y < f(x)\}.
\]

Since \( f \) is Lebesgue measurable, we know from Lemma 2.2 (iv) that \( f = \sup_n f_n \) for some increasing sequence \( f_n \) of bounded unsigned simple functions. Consider one such simple function \( f_n \), which may be assumed to take on \( N \) values \( c_i \), \( i = 1, \ldots, N \) on disjoint measurable sets \( T_i \). Let \( S_n = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : 0 \leq y < f_n(x)\} \). From Problem 1.21 (b) we know that \( S_n = \bigcup_{i=1}^{N} T_i \times [0, c_i] \) is measurable, with

\[
m(S_n) = \sum_{i=1}^{N} c_i m(T_i) = \int_{\mathbb{R}^d} f_n,
\]

using finite additivity. Since the sequence \( \{f_n\}_{n=1}^{\infty} \) is increasing, so is the corresponding sequence of sets \( S_n \). Also, we see that \( S = \bigcup_{n=1}^{\infty} S_n \). Upward monotone convergence then implies

\[
\lim_{n \to \infty} m(S_n) = m\left( \bigcup_{n=1}^{\infty} S_n \right) = m(S).
\]

Finally, since \( f \leq g \) implies \( \int_{\mathbb{R}^d} f \leq \int_{\mathbb{R}^d} g \), we conclude that

\[
\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} f = \sup_{h \text{ simple}} \int_{\mathbb{R}^d} h = \sup_n \int_{\mathbb{R}^d} f_n = \lim_{n \to \infty} \int_{\mathbb{R}^d} f_n = \lim_{n \to \infty} m(S_n) = m(S).
\]
6. Let $D \subset \mathbb{R}$ be measurable, and let $f : D \to [0, \infty]$ be measurable such that $\int_D f = 0$. Show that $f = 0$ a.e.

For $n \in \mathbb{N}$, set

$$D_n = \left\{ x \in D : f(x) \geq \frac{1}{n} \right\}.$$

Since

$$0 = \int_D f \geq \int_{D_n} \frac{1}{n} = \frac{1}{n} m(D_n),$$

we see that $m(D_n) = 0$ for every $n \in \mathbb{N}$. From subadditivity and upward monotonocity we then conclude that

$$0 \leq m(\{x \in D : f(x) \neq 0\}) = m\left(\bigcup_{n=1}^{\infty} D_n\right) \leq \sum_{i=1}^{\infty} m(D_n) = 0.$$

**Alternative solution:** The result follows directly on taking the limit $\lambda \to 0$ and using upward monotonocity in Markov’s inequality.

7. Recall that a $\sigma$-algebra is a Boolean algebra that is closed under countable unions. Let $X$ be a metric space and $\mathcal{F}$ be a collection of subsets of $X$. Denote the intersection of all $\sigma$-algebras that contain $\mathcal{F}$ as $\langle \mathcal{F} \rangle$. The Borel $\sigma$-algebra $\mathcal{B}[X]$ is that $\sigma$-algebra generated by the collection of open subsets of $X$. Families $\mathcal{F}_1$ and $\mathcal{F}_2$ generate the same $\sigma$-algebra if $\langle \mathcal{F}_1 \rangle \supset \langle \mathcal{F}_2 \rangle$ and $\langle \mathcal{F}_2 \rangle \supset \langle \mathcal{F}_1 \rangle$. Show that the collection of subsets of $\mathbb{R}^d$ that are:

- open;
- closed;
- compact;
- open balls;
- boxes;
- elementary

all generate the Borel $\sigma$-algebra on $\mathbb{R}^d$. Use the notation $\langle \text{open} \rangle$, $\langle \text{closed} \rangle$, etc.

- By definition, the Borel $\sigma$-algebra is the $\sigma$-algebra generated by the collection of open sets.
- Since a $\sigma$-algebra is closed under complements, $\langle \text{open} \rangle$ contains $\langle \text{closed} \rangle$ and $\langle \text{closed} \rangle$ contains $\langle \text{open} \rangle$.
- Since every compact set in $\mathbb{R}^d$ is closed, $\langle \text{closed} \rangle$ contains $\langle \text{compact} \rangle$.
- Every open set can be written as a countable union of open balls, so $\langle \text{open balls} \rangle$ contains $\langle \text{open} \rangle$. Since open balls are open, $\langle \text{open} \rangle$ contains $\langle \text{open balls} \rangle$. 
• Since every open set can be written as a countable union of almost disjoint closed cubes, \( \langle \text{boxes} \rangle \) contains \( \langle \text{open} \rangle \). Since closed cubes are compact, \( \langle \text{compact} \rangle \) contains \( \langle \text{open} \rangle \) and hence also \( \langle \text{boxes} \rangle \) (finite intersections of open and closed sets).

• Every elementary set is a finite union of boxes, so \( \langle \text{boxes} \rangle \) contains \( \langle \text{elementary} \rangle \). Since boxes are elementary, \( \langle \text{elementary} \rangle \) contains \( \langle \text{boxes} \rangle \).

There are many other possible routes to showing these equivalent representations of the Borel \( \sigma \)-algebra. For example, one can show directly that \( \langle \text{compact} \rangle \) contains \( \langle \text{closed} \rangle \) since every closed set \( F \) can written as a countable union of compact sets \( F \cap B_k[0] \).