1. What is wrong in the following argument?

Express the open set \((0, 1)\) as a union of open intervals \(I_x\) centered on each real \(x \in (0, 1)\):

\[
\bigcup_{x \in (0, 1)} I_x = (0, 1).
\]

Each interval \(I_x\) contains a rational number, which we can use as a label for that interval. Since there are only countably many rational numbers, there are only countably many labels. Hence the above union actually contains only a countable number of intervals, each centered at a real number. There are therefore only a countable number of real numbers.

While the labels are indeed countable, each label is shared by an uncountable number of intervals (each label does not uniquely identify an interval, so we cannot count intervals by counting labels).

2. Let \(S \subset \mathbb{R}^d\) such that

\[
m^*(E \cap S) + m^*(E \cap S^c) = m(E)
\]

for every elementary set \(E\). Show that \(S\) is Lebesgue measurable.

Let \(E\) be an elementary set.

If \(m^*(S)\) is finite, given \(\epsilon > 0\), there exists a countable family of disjoint boxes \(B_1, B_2, \ldots\) that cover \(S\) such that

\[
\sum_{k=1}^{\infty} |B_k| < m^*(S) + \epsilon.
\]

Each box \(B_k\) can be enclosed within an open box \(B'_k\) such that \(|B'_k| < |B_k| + \epsilon/2^k\). Since each \(B'_k\) is an elementary set,

\[
m^*(B'_k \cap S) + m^*(B'_k \cap S^c) = |B'_k|,
\]

from which we conclude that, using subadditivity:

\[
m^*(S) + m^*(U \cap S^c) \leq \sum_{k=1}^{\infty} [m^*(B'_k \cap S) + m^*(B'_k \cap S^c)] = \sum_{k=1}^{\infty} |B'_k| < m^*(S) + 2\epsilon.
\]

This implies that \(m^*(U \setminus S) \leq 2\epsilon\). Thus \(S\) is measurable.

If \(m^*(S) = \infty\), for each \(k \in \mathbb{N}\) let \(S_k = S \cap B_k \subset \mathbb{R}^d\), where \(B_k\) is a box of side \(k\) centered on zero. Note that

\[
E \cap S_k^c = (E \cap B_k^c) \cup (E \cap B_k \cap S^c).
\]
Using countable subadditivity and the fact that \( E \cap B_k \) is an elementary set, we find

\[
m(E) \leq m^*(E \cap S_k^c) + m^*(E \cap S_k) \\
\leq m(E \cap B_k^c) + m^*((E \cap B_k) \cap S^c) + m^*((E \cap B_k) \cap S) \\
= m(E \cap B_k^c) + m(E \cap B_k) = m(E)
\]

We thus have \( m^*(E \cap S_k^c) + m^*(E \cap S_k) = m(E) \) for every elementary set \( E \). But since \( S_k \) has finite measure, we have already proven that it is Lebesgue measurable. Hence the countable union \( S = \bigcup_{k=0}^{\infty} S_k \) is also Lebesgue measurable.

3. Suppose \( S_n \subset \mathbb{R}^d \), \( n = 1, 2, \ldots \) are Lebesgue measurable sets that converge pointwise to a set \( S \).

   (i) Show that \( S \) is Lebesgue measurable. Hint: use the fact that \( 1_S(x) = \lim \inf_{n \to \infty} 1_{S_n}(x) = \lim \sup_{n \to \infty} 1_{S_n}(x) \) to write \( S \) in terms of countable unions and intersections of \( S_n \).

By definition, we are given that \( \lim \inf_{n \to \infty} 1_{S_n}(x) = \lim \sup_{n \to \infty} 1_{S_n}(x) = 1_S(x) \).

This implies

\[
\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S_n = S.
\]

Being a countable union of a countable intersection of Lebesgue measurable sets, \( S \) is also Lebesgue measurable sets.

(ii) (Dominated convergence theorem) Suppose that the \( S_n \) are all contained in another Lebesgue measurable set \( F \) of finite measure. Show that \( m(S_n) \) converges to \( m(S) \). Hint: use the upward and downward monotone convergence theorems.

By monotonicity, each of the sets \( S_n \) has finite measure. Consider the decreasing sequence of sets \( T_k = \bigcap_{n=1}^{\infty} S_n \). By the downward monotone convergence theorem, \( m(T_k) = \lim_{n \to \infty} m(S_n) \) for each \( k \in \mathbb{N} \). Now consider the increasing sequence of sets \( U_n = \bigcup_{k=1}^{n} T_k = T_1 \). By the upward monotone convergence theorem,

\[
m(S) = m\left( \bigcup_{n=1}^{\infty} U_n \right) = \lim_{n \to \infty} m(U_n) = m(T_1) = \lim_{n \to \infty} m(S_n).
\]

(iii) Give a counterexample to show that the dominated convergence theorem fails if the \( S_n \) are not contained in a set of finite measure, even if we assume that the \( m(S_n) \) are all uniformly bounded.

Consider \( S_n = [n, n+1] \) for \( n \in \mathbb{N} \). Note that \( S_n \) converge pointwise to \( \emptyset \), yet

\[
\lim_{n \to \infty} m(S_n) = \lim_{n \to \infty} 1 = 1 \neq 0 = m(\emptyset).
\]
4. Show that an unsigned function \( f : \mathbb{R}^d \rightarrow [0, \infty] \) is a simple function iff it is measurable and takes on finitely many values.

If \( f \) is a simple function, it is a linear combination of \( n \) indicator functions, and therefore measurable. These indicator functions divide \( \mathbb{R}^d \) into at most \( 2^n \) regions. The number of values it can achieve is therefore at most \( 2^n \).

Conversely, if \( f \) is measurable and takes on finitely many values \( c_1, \ldots, c_n \), consider the Lebesgue measurable sets \( S_k = f^{-1}(\{c_k\}) \) for \( k = 1, \ldots, n \). We can then express \( f = \sum_{k=1}^{n} c_k 1_{S_k} \).

5. Let \( d, d' \in \mathbb{N} \).

   (a) If \( A \subset \mathbb{R}^d \) and \( B \subset \mathbb{R}^{d'} \), show that
   \[
m^{d+d'}(A \times B) \leq m^d(A)m^{d'}(B),
   \]
   where \( m^d \) denotes \( d \)-dimensional Lebesgue measure.

   Let \( \{A_i\} \) be a sequence of disjoint boxes such that \( A \subset \bigcup_{i=1}^{\infty} A_i \), with \( \sum_{i=1}^{\infty} |A_i| < m^*(A) + \epsilon \).

   Let \( \{B_j\} \) be a sequence of disjoint boxes such that \( B \subset \bigcup_{j=1}^{\infty} B_j \), with \( \sum_{j=1}^{\infty} |B_j| < m^*(B) + \epsilon \).

   Then \( A \times B \subset \bigcup_{i,j} A_i \times B_j \). Each \( A_i \times B_j \) is a box in \( \mathbb{R}^{d+d'} \) with measure \( |A_i \times B_j| = |A_i||B_j| \). By monotonicity,
   \[
m^{d+d'}(A \times B) \leq \sum_{i,j} |A_i| |B_j| \leq (m^d(A) + 2\epsilon)(m^{d'}(B) + 2\epsilon).
   \]

   Since this holds for all \( \epsilon > 0 \), the desired result follows.

   (b) Let \( A \subset \mathbb{R}^d \) and \( B \subset \mathbb{R}^{d'} \) be Lebesgue measurable sets (not necessarily of finite measure). Show that \( A \times B \) is Lebesgue measurable, with
   \[
m^{d+d'}(A \times B) = m^d(A)m^{d'}(B).
   \]

First we consider the case where \( A \) and \( B \) have finite measure. Given \( \epsilon > 0 \), let \( U, V \) be open sets such that \( m^*(U \setminus A) < \epsilon \) and \( m^*(V \setminus B) < \epsilon \). From subadditivity, we know that \( m^*(U) \leq m^*(U \setminus A) + m^*(A) < m^*(A) + \epsilon \) and \( m^*(V) \leq m^*(V \setminus B) + m^*(B) < m^*(B) + \epsilon \). Since
   \[
   U \times V \setminus (A \times B) = ((U \setminus A) \times V) \cup (U \times (V \setminus B)),
   \]
we see from subadditivity that
   \[
m^*(U \times V \setminus (A \times B)) \leq m^*((U \setminus A) \times V) + m^*(U \times (V \setminus B))
   \leq m^*(U \setminus A)m^*(V) + m^*(U)m^*(V \setminus B)
   < \epsilon m^*(V) + m^*(U) + \epsilon m^*(U)
   < \epsilon [m^*(A) + m^*(B) + 2\epsilon].
   \]
As $\epsilon$ is arbitrary, we see that $A \times B$ is Lebesgue measurable. Moreover, we know that the open sets $U = \bigcup_{i=1}^{\infty} U_i$ and $V = \bigcup_{i=1}^{\infty} V_i$ can be expressed as countable unions of almost disjoint boxes $U_i$ and $V_i$, respectively. Then

$$m(U \times V) = \sum_{i,j} |U_i||V_j| = \sum_i |U_i| \sum_j |V_j| = m(U)m(V)$$

Then from part (a), we find

$$m(A)m(B) - m(A \times B) = [m(U) - m(U \setminus A)][m(V) - m(V \setminus B)]$$

$$-m(U \times V) + m((U \times V) \setminus (A \times B))$$

$$= [m(U) - m(U \setminus A)][m(V) - m(V \setminus B)]$$

$$-m(U \times V) + m((U \setminus A) \times V) + m(U \times (V \setminus B))$$

$$= -m(U \setminus A)m(V) - m(U)m(V \setminus B)$$

$$+ m((U \setminus A) \times V) + m(U \times (V \setminus B))$$

$$< \epsilon^2.$$

Since $\epsilon$ is arbitrary, we see that

$$m(A)m(B) - m(A \times B) \leq 0.$$

On combining this inequality, with part (a), we obtain the desired result.

Finally, if $A$ or $B$ has infinite Lebesgue measure, first establish the result for $A \cap B_n(0)$ and $B \cap B_n(0)$ and then apply upward monotone convergence.

6. If $f : \mathbb{R}^d \to [0, \infty]$ is Lebesgue measurable, show that the Lebesgue measure of $\{(x, y) \in \mathbb{R}^d \times \mathbb{R} : 0 \leq y \leq f(x)\}$ exists and equals $\int_{\mathbb{R}^d} f$.

Let

$$S \triangleq \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : 0 \leq y \leq f(x)\}.$$

Since $f$ is Lebesgue measurable, we know from Lemma 1.14 (iv) that $f = \sup_n f_n$ for some increasing sequence $f_n$ of bounded unsigned simple functions. Consider one such simple function $f_n$, which may be assumed to take on $N$ values $c_i$, $i = 1, \ldots, N$ on distinct measurable sets $T_i$. Let $S_n \triangleq \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : 0 \leq y \leq f_n(x)\}$. From Question 5b we know that $S_n = \bigcup_{i=1}^{N} T_i \times [0, c_i]$ is measurable, with

$$m(S_n) = \sum_{i=1}^{n} c_i m(T_i) = \int_{\mathbb{R}^d} f_n,$$

using finite additivity. Since the sequence $\{f_n\}_{n=1}^{\infty}$ is increasing, so is the corresponding sequence of sets $S_n$. Also, we see that $S = \bigcup_{n=1}^{\infty} S_n$. Upward monotone convergence then implies

$$\lim_{n \to \infty} m(S_n) = m\left(\bigcup_{n=1}^{\infty} S_n\right) = m(S).$$
Finally, since \( f \leq g \) implies \( \int_{\mathbb{R}^d} f \leq \int_{\mathbb{R}^d} g \), we conclude that

\[
\int_{\mathbb{R}^d} f = \sup_{h \text{ simple}} \int_{\mathbb{R}^d} h = \sup_{0 \leq h \leq \sup_n f_n} \int_{\mathbb{R}^d} f_n = \lim_{n \to \infty} \int_{\mathbb{R}^d} f_n = \lim_{n \to \infty} \mu(S_n) = \mu(S).
\]