1. If \( S \) is an elementary set, prove directly from the definitions of the Jordan inner and outer measures and monotonicity of the elementary measure that \( m^*(S) \) and \( m^*(S) \) both equal the Lebesgue measure \( m(S) \).

Since \( S \) is one of the elementary sets that we need to consider in the supremum,

\[
m^*(S) = \sup_{E \subset S} m(E) \geq m(S).
\]

Also

\[
m^*(S) = \inf_{E \supset S} m(E) \leq m(S),
\]

so that \( m^*(S) \geq m^*(S) \). By monotonicity of the elementary measure, we also see that \( m^*(S) \leq m^*(S) \). Hence \( m^*(S) = m^*(S) = m(S) \).

2. Let \( \{B_k\}_{k=1}^{\infty} \) be a sequence of almost disjoint boxes. Prove that the Lebesgue measure of \( \bigcup_{k=1}^{\infty} B_k \) is equal to its Jordan inner measure. Hint: Use Question 5.

Let \( S = \bigcup_{k=1}^{\infty} B_k \). By countable additivity,

\[
m(S) = \sum_{k=1}^{\infty} |B_k|.
\]

For each \( n \in \mathbb{N} \), let \( S_n = \bigcup_{k=1}^{n} B_k \). Since \( S_n \subset S \), we know from the definition of Jordan inner measure that

\[
m^*(S) \geq m(S_n).
\]

From monotone upward convergence, we then see that

\[
m^*(S) \geq \lim_{n \to \infty} m(S_n) = m\left( \bigcup_{k=1}^{\infty} B_k \right) = m(S).
\]

But we have also seen that

\[
m^*(S) \leq m(S).
\]

Hence

\[
m^*(S) = m(S).
\]
3. Let $S \subset \mathbb{R}^d$. Prove that $S$ is Lebesgue measurable $\iff$
\[ m^*(E \cap S) + m^*(E \cap S^c) = m(E) \]
for every elementary set $E$.

\[ \Rightarrow \] Suppose $S \subset \mathbb{R}^d$ is Lebesgue measurable. Since for every elementary set $E$ both $E \cap S$ and $E \cap S^c$ are Lebesgue measurable, the finite additivity of Lebesgue measure implies that
\[ m(E \cap S) + m(E \cap S^c) = m(E). \]

\[ \Leftarrow \] If $m^*(S)$ is finite, given $\epsilon > 0$, there exists a countable family of disjoint boxes $B_1, B_2, \ldots$ that cover $S$ such that
\[ \sum_{k=1}^{\infty} |B_k| < m^*(S) + \epsilon. \]
Each box $B_k$ can be enclosed within an open box $B'_k$ such that $|B'_k| < |B_k| + \epsilon/2^k$. Since each $B'_k$ is an elementary set,
\[ m^*(B'_k \cap S) + m^*(B'_k \cap S^c) = |B'_k|. \]
Consider the open set $U = \bigcup_{k=1}^{\infty} B'_k$. Using subadditivity, we find
\[ m^*(S) + m^*(U \cap S^c) \leq \sum_{k=1}^{\infty} [m^*(B'_k \cap S) + m^*(B'_k \cap S^c)] = \sum_{k=1}^{\infty} |B'_k| < m^*(S) + 2\epsilon. \]
This implies that $m^*(U \setminus S) \leq 2\epsilon$. Thus $S$ is measurable.

If $m^*(S) = \infty$, for each $k \in \mathbb{N}$ let $S_k = S \cap B_k \subset \mathbb{R}^d$, where $B_k$ is a box of side $k$ centered on zero. Note that $S_k^c = B_k^c \cup (B_k \cap S^c)$. Let $E$ be an elementary set. Using subadditivity and the fact that $E \cap B_k$ is an elementary set, we find
\[ m(E) \leq m^*(E \cap S_k^c) + m^*(E \cap S_k) \]
\[ \leq m(E \cap B_k^c) + m^*(E \cap B_k) + m^*((E \cap B_k) \cap S^c) + m^*((E \cap B_k) \cap S) \]
\[ = m(E \cap B_k^c) + m(E \cap B_k) + m(E \cap B_k) \]
\[ = m(E). \]
We thus have $m^*(E \cap S_k^c) + m^*(E \cap S_k) = m(E)$ for every elementary set $E$. But since $S_k$ has finite measure, we have already proved that it is Lebesgue measurable. Hence the countable union $S = \bigcup_{k=0}^{\infty} S_k$ is also Lebesgue measurable.
4. Provide a counterexample that establishes that the claim

\[ m^*(S) = \sup_{\substack{U \subseteq S \\ U \text{ open}}} m^*(U) \]

is false. Justify your answer.

Since \( \mathbb{Q}^c \) and \([0,1]\) are Lebesgue measurable, so is their intersection \( S = \mathbb{Q}^c \cap [0,1] \).

\[ m(S) + m(\mathbb{Q} \cap [0,1]) = m([0,1]) = 1. \]

Since the countable set \( m(\mathbb{Q} \cap [0,1]) \) has measure zero, we see that \( m(S) = 1 \). However, \( \mathbb{Q} \) is dense in \([0,1]\), so the only open set contained within \( S \) is the empty set. Thus

\[ \sup_{\substack{U \subseteq S \\ U \text{ open}}} m^*(U) = m^*(\emptyset) = 0 \neq 1 = m^*(S). \]
5. Prove the monotone convergence theorem for measurable sets:

(i) *(Upward monotone convergence)* Let \( S_1 \subset S_2 \subset \ldots \) be a countable increasing sequence of Lebesgue measurable subsets of \( \mathbb{R}^d \). Show that \( \lim_{n \to \infty} m(S_n) = m(\bigcup_{k=1}^{\infty} S_k) \). Hint: Express \( \bigcup_{k=1}^{\infty} S_k \) in terms of the lacuna \( L_n = S_n \setminus \bigcup_{k=1}^{n-1} S_k \).

In terms of the lacuna \( L_n = S_n \setminus \bigcup_{k=1}^{n-1} S_k \), with \( L_1 = S_1 \), we can use induction (or Question 2 of Assignment 1) to express

\[
S_n = \bigcup_{k=1}^{n} S_k = \bigcup_{k=1}^{n} L_k,
\]
on noting that \( S_k = S_{k-1} \cup L_k \). Then since the sets \( L_k \) are disjoint,

\[
\lim_{n \to \infty} m(S_n) = \lim_{n \to \infty} m\left( \bigcup_{k=1}^{n} L_k \right) = \lim_{n \to \infty} \sum_{k=1}^{n} m(L_k) = m\left( \bigcup_{k=1}^{\infty} L_k \right) = m\left( \bigcup_{k=1}^{\infty} S_k \right).
\]

(ii) *(Downward monotone convergence)* Let \( S_1 \supset S_2 \supset \ldots \) be a countable decreasing sequence of Lebesgue measurable subsets of \( \mathbb{R}^d \). If at least one of the measures \( m(S_k) \) is finite, show that \( m\left( \bigcap_{k=1}^{\infty} S_k \right) = \lim_{n \to \infty} m(S_n) \).

Since neither the countable intersection of \( S_k \) nor the limit of \( m(S_n) \) are affected if we remove a finite number of the sets \( S_k \), we may assume without loss of generality that the measure of the first set \( S_1 \) is finite. Let \( T_n = S_1^c \cap S_1 \) for \( n \in \mathbb{N} \). From the additivity of Lebesgue measurable sets, we know that

\[
m\left( \bigcup_{k=1}^{\infty} T_k \right) = m\left( \bigcup_{k=1}^{\infty} T_k \cap S_1 \right) = m(S_1),
\]
or equivalently,

\[
m\left( \bigcap_{k=1}^{\infty} S_k \right) + m\left( \bigcup_{k=1}^{\infty} T_k \right) = m(S_1).
\]

Likewise,

\[
m(S_n) + m(T_n) = m(S_1),
\]

On applying upward monotone convergence to the increasing sequence \( T_1 \subset T_2 \subset \ldots \), we deduce

\[
m(S_1) - m\left( \bigcap_{k=1}^{\infty} S_k \right) = m\left( \bigcup_{k=1}^{\infty} T_k \right) = \lim_{n \to \infty} m(T_n) = m(S_1) - \lim_{n \to \infty} m(S_n).
\]

Since \( m(S_1) \) is finite, this implies that \( m\left( \bigcap_{k=1}^{\infty} S_k \right) = \lim_{n \to \infty} m(S_n) \).
(iii) Show that one cannot drop the assumption in (ii) that at least one of the measures $m(S_k)$ is finite.

Consider the decreasing sequence of sets $S_n = B_n(0)^c$ for $n \in \mathbb{N}$. Then

$$m \left( \bigcap_{k=1}^{\infty} S_k \right) = m(\emptyset) = 0$$

but

$$\lim_{n \to \infty} m(S_n) = \infty.$$ 

6. (Inner regularity). Let $S \subset \mathbb{R}^d$ be Lebesgue measurable. Show that

$$m(S) = \sup_{K \subset S, K \text{ compact}} m(K).$$

If $S$ is bounded, we know from Lemma 1.11, that there exists a closed and bounded set $K \subset S \subset \mathbb{R}^d$, with $m(K) > m(S) - \epsilon$.

Otherwise, let $S_k = S \cap B_k(0) \subset \mathbb{R}^d$ for $k = 1, 2, \ldots$, so that

$$S = \bigcup_{k=0}^{\infty} S_k$$

and $S_1 \subset S_2 \subset \ldots$. The monotone convergence theorem implies that

$$\lim_{n \to \infty} m(S_n) = m \left( \bigcup_{k=0}^{\infty} S_k \right) = m(S).$$

From Lemma 1.11, we know for every $\epsilon > 0$ that there exists for each $n \in \mathbb{N}$ a compact set $K_n \subset S_n \subset S$, with $m(K_n) > m(S_n) - \epsilon$ (since each subset $S_n$ is bounded).

If $m(S) = \infty$, choose $\epsilon = 1$, so that $m(K_n) > m(S_n) - 1$. Since $\lim_{n \to \infty} m(S_n) = \infty$, we see that $\lim_{n \to \infty} m(K_n) = \infty$. Hence

$$\sup_{K \subset S, K \text{ compact}} m(K) = \infty.$$ 

If $m(S) < \infty$, note that

$$m \left( \bigcup_{k=1}^{\infty} S \cap [B_k(0) \setminus B_{k-1}(0)] \right) = m(S) < \infty.$$ 

Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$m(S_N) = \sum_{k=1}^{N} m(S \cap [B_k(0) \setminus B_{k-1}(0)]) > m(S) - \epsilon.$$ 

Since $S_N$ is bounded, there exists a compact set $K_N \subset S_N \subset S$ with $m(K_N) > m(S_N) - \epsilon > m(S) - 2\epsilon$. Hence

$$\sup_{K \subset S, K \text{ compact}} m(K) = m(S).$$