Math 417: Honours Real Variables I  
Winter, 2019 Assignment 1  
January 18, due January 28

1. Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of sets. Define

$$
\lim\inf_{n \to \infty} S_n \doteq \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S_n
$$

and

$$
\lim\sup_{n \to \infty} S_n \doteq \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n.
$$

Show that

$$
\lim\inf_{n \to \infty} S_n \subset \lim\sup_{n \to \infty} S_n.
$$

Let $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S_n$. By the definition of a union

$$
x \in \bigcap_{n=m}^{\infty} S_n
$$

for some $m \in \mathbb{N}$. Hence, by the definition of an intersection, for every $n \geq m$ we have $x \in S_n$. Then certainly for every $k \in \mathbb{N}$,

$$
x \in \bigcup_{n=k}^{\infty} S_n.
$$

Thus $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n$.

2. Let $\{S_j\}_{j=1}^{\infty}$ be a sequence of sets. Define $T_1 = S_1$ and

$$
T_j = S_j \setminus \bigcup_{i=1}^{j-1} S_i, \quad j = 2, 3, \ldots
$$

Prove that $\{T_j\}_{j=1}^{\infty}$ is a sequence of pairwise disjoint sets satisfying

$$
\bigcup_{j=1}^{n} T_j = \bigcup_{j=1}^{n} S_j, \quad \forall n \in \mathbb{N}.
$$

By definition, $T_j \subset S_j$ and $T_j \cap S_i = \emptyset$ whenever $i < j$. This means that

$$
T_j \cap T_i \subset T_j \cap S_i = \emptyset \quad \text{whenever} \ i < j.
$$
That is, the sets $T_j$ are pairwise disjoint. Since $T_j \subset S_j$, we see that $\bigcup_{j=1}^n T_j \subset \bigcup_{j=1}^n S_j$. Conversely, for $x \in \bigcup_{j=1}^n S_j$, let $m$ be the smallest integer for which $x \in S_m$. Then

$$x \in T_m \subset \bigcup_{j=1}^n T_j.$$ 

Thus

$$\bigcup_{j=1}^n S_j \subset \bigcup_{j=1}^n T_j.$$ 

3. Suppose that $A$ and $B$ are subsets of $\mathbb{R}^d$, with $m^*(A) < \infty$ and $m^*(B) < \infty$. Prove that

$$|m^*(A) - m^*(B)| \leq m^*(A \Delta B).$$

Since $A \subset (A \setminus B) \cup B \subset (A \Delta B) \cup B$, it follows that

$$m^*(A) \leq m^*(B \cup (A \Delta B)) \leq m^*(B) + m^*(A \Delta B),$$

which together with the assumption $m^*(B) < \infty$ implies

$$m^*(A) - m^*(B) \leq m^*(A \Delta B).$$

On interchanging $A$ and $B$ we find

$$m^*(B) - m^*(A) \leq m^*(A \Delta B).$$

The desired result then follows.

4. Give examples of

(a) a bounded countable union of Jordan measurable sets that is not Jordan measurable;

Let $\{q_1, q_2, \ldots\}$ be an enumeration of $\mathbb{Q} \cap [0, 1]$, which we can write as a countable union of sets of Jordan measure zero:

$$\bigcup_{k=1}^{\infty} \{q_k\}.$$ 

(b) a bounded countable intersection of Jordan measurable sets that is not Jordan measurable.

The set of irrational numbers in $[0, 1]$ can be written as

$$\bigcap_{k=1}^{\infty} [0, 1] \setminus \{q_k\}.$$
5. Determine whether each of the following sets is countable. Justify your answers.

(a) The set of all mappings from \{1, \ldots, N\} to \N, where \N is a fixed positive integer.
   For \(N = 2\), the number of mappings from \{1, 2\} to \N is just the cardinality of \(N^2 = N \times N\), which we know is countable (e.g. \Q is countable). Suppose that \(N^N\) is countable for \(N \geq 2\). Then \(N^{N+1} = N^N \times N\) is isomorphic to \(N \times N\), which again is countable. By induction, we see that these sets are countable for every \(N \in \N\).

(b) The set of all mappings from \N to \[0, 1\].
   Since there are already as many ways to map 1 to a real number in \[0, 1\] as there are real numbers in \[0, 1\], this set is uncountable.

(c) The set of all mappings \(f\) from \N to \[0, 1\] that are “eventually zero”; i.e. there is a positive integer \(N\) such that \(f(n) = 0\) for all \(n > N\).
   For \(n > 1\), we see that there are already as many ways to map 1 to a real number in \[0, 1\] as there are real numbers in \[0, 1\], so this set is also uncountable.

(d) The set of all finite subsets of \N.
   Given a finite subset \(F\) of \N, let \(b_k = 1\) if \(k \in F\) and \(b_k = 0\) otherwise. Then \(\sum_{k=1}^{\infty} b_k 2^{-k}\) is a rational number in \[0, 1\]. Since the set of all rational numbers is countable, so is the set of all finite subsets of \N.
   Alternatively, since \N is countable, this follows from Question 8.

6. If \(\{x_\alpha\}_{\alpha \in A}\) is a collection of numbers \(x_\alpha \in [0, \infty]\) such that \(\sum_{\alpha \in A} x_\alpha < \infty\), show that \(x_\alpha = 0\) for all but at most countably many \(\alpha \in A\), even if \(A\) is itself uncountable.

Let

\[ S = \sup_{F \in A, \text{finite}} \sum_{\alpha \in F} x_\alpha. \]

For each \(n \in \N\), there exists a finite subset \(F_n \subset A\) such that

\[ S - \frac{1}{n} < \sum_{\alpha \in F_n} x_\alpha \leq S. \]

Consider the countable set \(T = \bigcup_{n=1}^{\infty} F_n\). We see that

\[ \sum_{\alpha \in T} x_\alpha = S. \]

Since the numbers \(x_\alpha\) are all non-negative we conclude that \(x_\alpha = 0\) for all \(\alpha \notin T\).

7. Let \(S\) be a set. Prove that there is no surjective map from \(f : S \mapsto \mathcal{P}(S)\). Hint: Consider the set \(T = \{s \in S : s \notin f(s)\}\).

As \(f\) is surjective there exists \(t \in S\) such that \(f(t) = T\). If \(t \in T\), this contradicts \(t \notin f(t) = T\). Likewise, \(t \notin T\) contradicts \(t \in f(t) = T\).
8. Let $S$ be a countable set. Show that

$$\mathcal{F}(S) \doteq \{ F \subseteq S : F \text{ is finite} \}$$

is a countable subset of $\mathcal{P}(S)$.

For $n \in \mathbb{N}$, set

$$\mathcal{F}_n(S) \doteq \{ F \subseteq S : \#F = n \},$$

so that $\mathcal{F}(S) = \emptyset \cup \bigcup_{n=1}^{\infty} \mathcal{F}_n(S)$. The union of countably many countable sets is again countable. Hence, it is sufficient to show for each $n \in \mathbb{N}$ that $\mathcal{F}_n(S)$ is countable.

Fix $n \in \mathbb{N}$. The Cartesian product of two countable sets is again countable; inductively, we conclude that the $n$-fold Cartesian product $S^n$ is countable. Consider

$$q : S^n \to \mathcal{P}(S), \quad (x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\}.$$

We observe that $\mathcal{F}_n(S) = q(S^n)$ is indeed countable.

Alternatively, since $S$ is isomorphic to $\mathbb{N}$, this follows from Question 5(d).