Math 411: Honours Complex Variables
List of Theorems

Theorem 1.1 (C is a Field). The complex numbers are a field. Specifically, we have:

• $(0, 0)$ is the identity element of addition;
• $-(x, y) = (-x, -y)$ for $x, y \in \mathbb{R}$;
• $(1, 0)$ is the identity element of multiplication;
• $(x, y)^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$ for $x, y \in \mathbb{R}$ with $(x, y) \neq (0, 0)$.

Theorem 2.1 (Cauchy–Riemann Equations). Let $D \subset \mathbb{C}$ be open, and let $z_0 \in D$. Let $f : D \to \mathbb{C}$ and denote $u := \text{Re } f$, $v := \text{Im } f$. Then the following are equivalent:

(i) $f$ is complex differentiable at $z_0$;

(ii) $f$ is totally differentiable at $z_0$ (in the sense of multivariable calculus), and the Cauchy–Riemann differential equations

\[
\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0)
\]

hold.

Corollary 2.1.1. Let $D \subset \mathbb{C}$ be open and connected, and let $f : D \to \mathbb{C}$ be complex differentiable. Then $f$ is constant on $D$ if and only if $f' \equiv 0$.

Theorem 3.1 (Radius of Convergence). Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a complex power series. Then there exists a unique $R \in [0, \infty)$ with the following properties:

• $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges absolutely for each $z \in B_R(z_0)$;
• for each $r \in [0, R)$, the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges uniformly on $B_r[z_0] := \{z \in \mathbb{C} : |z - z_0| \leq r\}$;
• $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ diverges for each $z \notin B_R[z_0]$.
Moreover, $R$ can be computed via the Cauchy–Hadamard formula:

$$R = \frac{1}{\limsup_{n \to \infty} \sqrt[n]{|a_n|}}.$$  

It is called the radius of convergence for $\sum_{n=0}^{\infty} a_n(z - z_0)^n$.

**Theorem 3.2** (Term-by-Term Differentiation). Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a complex power series with radius of convergence $R$. Then

$$f: B_R(z_0) \to \mathbb{C}, \quad z \mapsto \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

is complex differentiable at each point $z \in B_R(z_0)$ with

$$f'(z) = \sum_{n=1}^{\infty} n a_n(z - z_0)^{n-1}.$$  

**Corollary 3.2.1** (Higher Derivatives of Power Series). Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a complex power series with radius of convergence $R$. Then

$$f: B_R(z_0) \to \mathbb{C}, \quad z \mapsto \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

is infinitely often complex differentiable on $B_R(z_0)$ with

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1)a_n(z - z_0)^{n-k}.$$  

for $z \in B_R(z_0)$ and $k \in \mathbb{N}$. In particular, when $z = z_0$ we see that

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

holds for each $n \in \mathbb{N}_0$.

**Corollary 3.2.2** (Integration of Power Series). Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a complex power series with radius of convergence $R$. Then

$$F: B_R(z_0) \to \mathbb{C}, \quad z \mapsto \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}$$
is complex differentiable on $B_R(z_0)$ with

$$F'(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

for $z \in B_R(z_0)$.

**Theorem 4.1** (Antiderivative Theorem). Let $D \subset \mathbb{C}$ be open and connected and let $f : D \to \mathbb{C}$ be continuous. Then the following are equivalent:

(i) $f$ has an antiderivative;

(ii) $\int_\gamma f(\zeta) \, d\zeta = 0$ for any closed, piecewise smooth curve $\gamma$ in $D$;

(iii) for any piecewise smooth curve $\gamma$ in $D$, the value of $\int_\gamma f$ depends only on the initial point and the endpoint of $\gamma$.

**Theorem 5.1** (Goursat’s Lemma). Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $\Delta \subset D$ be a triangle. Then we have

$$\int_{\partial \Delta} f(\zeta) \, d\zeta = 0.$$ 

**Theorem 5.2.** Let $D \subset \mathbb{C}$ be open and star shaped with center $z_0$, and let $f : D \to \mathbb{C}$ be continuous such that

$$\int_{\partial \Delta} f(\zeta) \, d\zeta = 0$$

for each triangle $\Delta \subset D$ with $z_0$ as a vertex. Then $f$ has an antiderivative.

**Corollary 5.2.1.** Let $D \subset \mathbb{C}$ be open and star shaped, and let $f : D \to \mathbb{C}$ be holomorphic. Then $f$ has an antiderivative.

**Corollary 5.2.2.** Let $D \subset \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ be holomorphic. Then, for each $z_0 \in D$, there exists an open neighbourhood $U \subset D$ of $z_0$ such that $f|_U$ has an antiderivative.

**Corollary 5.2.3** (Cauchy’s Integral Theorem for Star-Shaped Domains). Let $D \subset \mathbb{C}$ be open and star shaped, and let $f : D \to \mathbb{C}$ be holomorphic. Then $\int_\gamma f(\zeta) \, d\zeta = 0$ holds for each closed curve $\gamma$ in $D$. 

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Theorem 5.3 (Cauchy’s Integral Formula for Circles). Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $z_0 \in D$ and $r > 0$ be such that $B_r[z_0] \subset D$. Then we have

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

for all $z \in B_r(z_0)$.

Corollary 5.3.1 (Mean Value Equation). Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $z_0 \in D$ and $r > 0$ be such that $B_r[z_0] \subset D$. Then we have

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) \, dt.$$ 

Theorem 5.4 (Higher Derivatives of Holomorphic Functions). Let $D \subset \mathbb{C}$ be open, let $z_0 \in D$ and $r > 0$ be such that $B_r[z_0] \subset D$, and let $f : D \to \mathbb{C}$ be continuous such that

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

holds for all $z \in B_r(z_0)$. Then $f$ is infinitely often complex differentiable on $B_r(z_0)$ and satisfies

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta$$  \hfill (*)

holds for all $z \in B_r(z_0)$ and $n \in \mathbb{N}_0$.

Corollary 5.4.1 (Generalized Cauchy Integral Formula). Let $D \subset \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ be holomorphic. Then $f$ is infinitely often complex differentiable on $D$. Moreover, for any $z_0 \in D$ and $r > 0$ such that $B_r[z_0] \subset D$, the generalized Cauchy integral formula holds, i.e.

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta$$

for all $z \in B_r(z_0)$ and $n \in \mathbb{N}_0$.

Theorem 5.5 (Characterizations of Holomorphic Functions). Let $D \subset \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ be continuous. Then the following are equivalent:
(i) \( f \) is holomorphic;

(ii) the Morera condition holds, i.e. \( \int_{\partial \Delta} f(\zeta) \, d\zeta = 0 \) for each triangle \( \Delta \subset D \);

(iii) for each \( z_0 \in D \) and \( r > 0 \) with \( B_r[z_0] \subset D \), we have
\[
f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta
\]
for \( z \in B_r(z_0) \);

(iv) for each \( z_0 \in D \), there exists \( r > 0 \) with \( B_r[z_0] \subset D \) and
\[
f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta
\]
for \( z \in B_r(z_0) \);

(v) \( f \) is infinitely often complex differentiable on \( D \);

(vi) for each \( z_0 \in D \), there exists an open neighbourhood \( U \subset D \) of \( z_0 \) such that \( f \) has an antiderivative on \( U \).

**Theorem 5.6** (Liouville’s Theorem). Let \( f : \mathbb{C} \to \mathbb{C} \) be a bounded entire function. Then \( f \) is constant.

**Corollary 5.6.1** (Fundamental Theorem of Algebra). Let \( p \) be a non-constant polynomial with complex coefficients. Then \( p \) has a zero.

**Theorem 6.1** (Uniform Convergence Preserves Continuity). Let \( D \subset \mathbb{C} \) be open, and let \( (f_n)_{n=1}^{\infty} \) be a sequence of continuous, \( \mathbb{C} \)-valued functions on \( D \) converging uniformly on \( D \) to \( f : D \to \mathbb{C} \). Then \( f \) is continuous.

**Theorem 6.2** (Weierstraß Theorem). Let \( D \subset \mathbb{C} \) be open, let \( f_1, f_2, \ldots : D \to \mathbb{C} \) be holomorphic such that \( (f_n)_{n=1}^{\infty} \) converges to \( f : D \to \mathbb{C} \) compactly. Then \( f \) is holomorphic, and \( (f_n^{(k)})_{n=1}^{\infty} \) converges compactly to \( f^{(k)} \) for each \( k \in \mathbb{N} \).

**Theorem 6.3** (Power Series for Holomorphic Functions). Let \( D \subset \mathbb{C} \) be open. Then the following are equivalent for \( f : D \to \mathbb{C} \):

(i) \( f \) is holomorphic;

(ii) for each \( z_0 \in D \), there exists \( r > 0 \) with \( B_r(z_0) \subset D \) and \( a_0, a_1, a_2, \ldots \in \mathbb{C} \) such that \( f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \) for all \( z \in B_r(z_0) \);

(iii) for each \( z_0 \in D \) and \( r > 0 \) with \( B_r(z_0) \subset D \), we have

\[
f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n
\]

for all \( z \in B_r(z_0) \).

**Theorem 7.1** (Identity Theorem). Let \( D \subset \mathbb{C} \) be open and connected, and let \( f, g : D \to \mathbb{C} \) be holomorphic. Then the following are equivalent:

(i) \( f = g \);

(ii) the set \( \{ z \in D : f(z) = g(z) \} \) has a cluster point in \( D \);

(iii) there exists \( z_0 \in D \) such that \( f^{(n)}(z_0) = g^{(n)}(z_0) \) for all \( n \in \mathbb{N}_0 \).

**Theorem 7.2** (Open Mapping Theorem). Let \( D \subset \mathbb{C} \) be open and connected, and let \( f : D \to \mathbb{C} \) be holomorphic and not constant. Then \( f(D) \subset \mathbb{C} \) is open and connected.

**Theorem 7.3** (Maximum Modulus Principle). Let \( D \subset \mathbb{C} \) be open and connected, and let \( f : D \to \mathbb{C} \) be holomorphic such that the function

\[
|f| : D \rightarrow \mathbb{C}, \quad z \mapsto |f(z)|
\]

attains a local maximum on \( D \). Then \( f \) is constant.

**Corollary 7.3.1.** Let \( D \subset \mathbb{C} \) be open and connected, and let \( f : D \to \mathbb{C} \) be holomorphic such that \( |f| \) attains a local minimum on \( D \). Then \( f \) is constant or \( f \) has a zero.

**Corollary 7.3.2** (Maximum Modulus Principle for Bounded Domains). Let \( D \subset \mathbb{C} \) be open, connected, and bounded, and let \( f : \overline{D} \to \mathbb{C} \) be continuous such that \( f|_D \) is holomorphic. Then \( |f| \) attains its maximum over \( \overline{D} \) on \( \partial D \).
Theorem 7.4 (Schwarz’s Lemma). Let $f : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ be holomorphic such that $f(0) = 0$. Then one has

$$|f(z)| \leq |z| \quad \text{for } z \in \mathbb{D} \quad \text{and} \quad |f'(0)| \leq 1.$$  

Moreover, if there exists $z_0 \in \mathbb{D} \setminus \{0\}$ such that $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$, then there exists $c \in \mathbb{C}$ with $|c| = 1$ such that $f(z) = cz$ for $z \in \mathbb{D}$.

Corollary 7.4.1. Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be biholomorphic such that $f(0) = 0$. Then there exists $c \in \mathbb{C}$ with $|c| = 1$ such that $f(z) = cz$ for $z \in \mathbb{D}$.

Theorem 7.5 (Biholomorphisms of $\mathbb{D}$). Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be biholomorphic. Then there exist $w \in \mathbb{D}$ and $c \in \partial \mathbb{D}$ with $f(z) = c\phi_w(z)$ for $z \in \mathbb{D}$.

Theorem 7.6 (Riemann’s Removability Condition). Let $D \subset \mathbb{C}$ be open, let $f : D \rightarrow \mathbb{C}$ be holomorphic, and let $z_0 \in \mathbb{C} \setminus D$ be an isolated singularity for $f$. Then the following are equivalent:

(i) $z_0$ is removable;

(ii) there is a continuous function $g : D \cup \{z_0\} \rightarrow \mathbb{C}$ such that $g|_D = f$;

(iii) there exists $\epsilon > 0$ with $B_\epsilon(z_0) \setminus \{z_0\} \subset D$ such that $f$ is bounded on $B_\epsilon(z_0) \setminus \{z_0\}$.

Theorem 8.1 (Poles). Let $D \subset \mathbb{C}$ be open, let $f : D \rightarrow \mathbb{C}$ be holomorphic, and let $z_0 \in \mathbb{C} \setminus D$ be an isolated singularity of $f$. Then $z_0$ is a pole of $f \iff$ there exist a unique $k \in \mathbb{N}$ and a holomorphic function $g : D \cup \{z_0\} \rightarrow \mathbb{C}$ such that $g(z_0) \neq 0$ and

$$f(z) = \frac{g(z)}{(z - z_0)^k}$$

for $z \in D$.

Theorem 8.2 (Casorati–Weierstraß Theorem). Let $D \subset \mathbb{C}$ be open, let $f : D \rightarrow \mathbb{C}$ be holomorphic, and let $z_0 \in \mathbb{C} \setminus D$ be an isolated singularity of $f$. Then $z_0$ is essential $\iff$ $f(B_\epsilon(z_0) \cap D) = \mathbb{C}$ for each $\epsilon > 0$.

Theorem 9.1 (Cauchy’s Integral Theorem for Annuli). Let $z_0 \in \mathbb{C}$, let $r, \rho, P, R \in [0, \infty]$ be such that $r < \rho < P < R$, and let $f : A_{r,R}(z_0) \rightarrow \mathbb{C}$ be holomorphic. Then we have

$$\int_{\partial B_{\rho}(z_0)} f(\zeta) \, d\zeta = \int_{\partial B_{\rho}(z_0)} f(\zeta) \, d\zeta.$$
Theorem 9.2 (Laurent Decomposition). Let \( z_0 \in \mathbb{C} \), let \( r, R \in [0, \infty) \) be such that \( r < R \), and let \( f: A_{r,R}(z_0) \to \mathbb{C} \) be holomorphic. Then there exists a holomorphic function

\[
g: B_R(z_0) \to \mathbb{C} \quad \text{and} \quad h: \mathbb{C} \setminus B_r[z_0] \to \mathbb{C}
\]

with \( f = g + h \) on \( A_{r,R}(z_0) \). Moreover, \( h \) can be chosen such that \( \lim_{|z| \to \infty} h(z) = 0 \), in which case \( g \) and \( h \) are uniquely determined.

Theorem 9.3 (Laurent Coefficients). Let \( z_0 \in \mathbb{C} \), let \( r, R \in [0, \infty) \) be such that \( r < R \), and let \( f: A_{r,R}(z_0) \to \mathbb{C} \) be holomorphic. Then \( f \) has a representation

\[
f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n
\]

for \( z \in A_{r,R}(z_0) \) as a Laurent series, which converges uniformly and absolutely on compact subsets of \( A_{r,R}(z_0) \). Moreover, for every \( n \in \mathbb{Z} \) and \( \rho \in (r, R) \), the coefficients \( a_n \) are uniquely determined as

\[
a_n = \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta.
\]

Corollary 9.3.1. Let \( z_0 \in \mathbb{C} \), let \( r > 0 \), and let \( f: B_r(z_0) \setminus \{z_0\} \to \mathbb{C} \) be holomorphic with Laurent representation \( f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n \). Then the singularity \( z_0 \) of \( f \) is

(i) removable if and only if \( a_n = 0 \) for \( n < 0 \);

(ii) a pole of order \( k \in \mathbb{N} \) if and only if \( a_{-k} \neq 0 \) and \( a_n = 0 \) for all \( n < -k \);

(iii) essential if and only if \( a_n \neq 0 \) for infinitely many \( n < 0 \).

Proposition 10.1. Let \( \gamma \) be a closed curve in \( \mathbb{C} \), and let \( z \in \mathbb{C} \setminus \{\gamma\} \). Then \( \nu(\gamma, z) \in \mathbb{Z} \).

Proposition 10.2 (Winding Numbers Are Locally Constant). Let \( \gamma \) be a closed curve in \( \mathbb{C} \). Then:

(i) the map

\[
\mathbb{C} \setminus \{\gamma\} \to \mathbb{C}, \quad z \mapsto \nu(\gamma, z)
\]

is locally constant.
(ii) there exists $R > 0$ such that $\mathbb{C} \setminus B_R[0] \subset \text{ext } \gamma$.

**Theorem 11.1** (Cauchy’s Integral Formula). Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $\gamma$ be a closed curve in $D$ that is homologous to zero. Then, for $n \in \mathbb{N}_0$ and $z \in D \setminus \{\gamma\}$, we have

$$\nu(\gamma, z) f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

**Theorem 11.2** (Cauchy’s Integral Theorem). Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $\gamma$ be a closed curve in $D$ that is homologous to zero. Then

$$\oint_{\gamma} f(\zeta) d\zeta = 0.$$

**Corollary 11.2.1.** Let $D$ be an open, connected subset of $\mathbb{C}$. Then $D$ is simply connected $\iff$ every holomorphic function on $D$ has an antiderivative.

**Corollary 11.2.2** (Holomorphic Logarithms). A simply connected domain admits holomorphic logarithms.

**Corollary 11.2.3** (Holomorphic Roots). A simply connected domain admits holomorphic roots.

**Theorem 12.1** (Residue Theorem). Let $D \subset \mathbb{C}$ be open and simply connected, $z_1, \ldots, z_n \in D$ be such that $z_j \neq z_k$ for $j \neq k$, $f : D \setminus \{z_1, \ldots, z_n\} \to \mathbb{C}$ be holomorphic, and $\gamma$ be a closed curve in $D \setminus \{z_1, \ldots, z_n\}$. Then we have

$$\int_{\gamma} f(\zeta) d\zeta = 2\pi i \sum_{j=1}^{n} \nu(\gamma, z_j) \text{res}(f, z_j).$$

**Corollary 12.1.1.** Let $D \subset \mathbb{C}$ be open and simply connected, $f : D \to \mathbb{C}$ be holomorphic, and $\gamma$ be a closed curve in $D$. Then we have

$$\nu(\gamma, z) f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$$

for $z \in D \setminus \{\gamma\}$.

**Proposition 12.1** (Rational Trigonometric Polynomials). Let $p$ and $q$ be polynomials of two real variables such that $q(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$ with $x^2 + y^2 = 1$. Then we have

$$\int_0^{2\pi} \frac{p(\cos t, \sin t)}{q(\cos t, \sin t)} dt = 2\pi i \sum_{z \in \mathbb{D}} \text{res}(f, z),$$
where

\[ f(z) = \frac{1}{iz} \cdot \frac{p \left( \frac{1}{2} \left( \frac{z + 1}{z} \right) , \frac{1}{2i} \left( \frac{z - 1}{z} \right) \right)}{q \left( \frac{1}{2} \left( \frac{z + 1}{z} \right) , \frac{1}{2i} \left( \frac{z - 1}{z} \right) \right)}. \]

**Proposition 12.2** (Rational Functions). Let \( p \) and \( q \) be polynomials of one real variable with \( \text{deg} \ q \geq \text{deg} \ p + 2 \) and such that \( q(x) \neq 0 \) for \( x \in \mathbb{R} \). Then we have

\[ \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \, dx = 2\pi i \sum_{z \in \mathbb{H}} \text{res} \left( \frac{p}{q} , z \right), \]

where

\[ \mathbb{H} := \{ z \in \mathbb{C} : \text{Im} \ z > 0 \}. \]

**Theorem 13.1** (Meromorphic Functions Form a Field). Let \( D \subset \mathbb{C} \) be open and connected. Then the meromorphic functions on \( D \), where we define \((f + g)(z) = \lim_{w \to z} [f(w) + g(w)]\) and \((fg)(z) = \lim_{w \to z} [f(w)g(w)]\), form a field.

**Theorem 13.2** (Argument Principle). Let \( D \subset \mathbb{C} \) be open and simply connected, let \( f \) be meromorphic on \( D \), and let \( \gamma \) be a closed curve in \( D \setminus (P(f) \cup Z(f)) \). Then we have

\[ \frac{1}{2\pi i} \int_{\gamma} \frac{f'(\zeta)}{f(\zeta)} \, d\zeta = \sum_{z \in Z(f)} \nu(\gamma, z) \text{ord}(f, z) - \sum_{z \in P(f)} \nu(\gamma, z) \text{ord}(f, z). \]

**Theorem 13.3** (Bifurcation Theorem). Let \( D \subset \mathbb{C} \) be open, let \( f : D \to \mathbb{C} \) be holomorphic, and suppose that, at \( z_0 \in D \), the function \( f \) attains \( w_0 \) with multiplicity \( k \in \mathbb{N} \). Then there exist neighbourhoods \( V \subset D \) of \( z_0 \) and \( W \subset f(V) \) of \( w_0 \) such that, for each \( w \in W \setminus \{w_0\} \), there exist distinct \( z_1, \ldots , z_k \in V \) with \( f(z_1) = \cdots = f(z_k) = w \), where \( f \) attains \( w \) at each \( z_j \) with multiplicity one.

**Theorem 13.4** (Hurwitz’s Theorem). Let \( D \subset \mathbb{C} \) be open and connected, let \( f, f_1, f_2, \ldots : D \to \mathbb{C} \) be holomorphic such that \((f_n)_{n=1}^{\infty}\) converges to \( f \) compactly on \( D \), and suppose that \( Z(f_n) = \emptyset \) for \( n \in \mathbb{N} \). Then \( f \equiv 0 \) or \( Z(f) = \emptyset \).

**Corollary 13.4.1.** Let \( D \subset \mathbb{C} \) be open and connected, let \( f, f_1, f_2, \ldots : D \to \mathbb{C} \) be holomorphic such that \((f_n)_{n=1}^{\infty}\) converges to \( f \) compactly on \( D \), and suppose that \( f_n \) is injective for \( n \in \mathbb{N} \). Then \( f \) is constant or injective.
Theorem 13.5 (Rouché’s Theorem). Let \( D \subset \mathbb{C} \) be open and simply connected, and let \( f, g : D \to \mathbb{C} \) be holomorphic. Suppose that \( \gamma \) is a closed curve in \( D \) such that \( \text{int} \gamma = \{ z \in D \setminus \{ \gamma \} : \nu(\gamma, z) = 1 \} \) and that

\[
|f(\zeta) - g(\zeta)| < |f(\zeta)|
\]

for \( \zeta \in \{ \gamma \} \). Then \( f \) and \( g \) have the same number of zeros in \( \text{int} \gamma \) (counting multiplicity).

Corollary 13.5.1 (Fundamental Theorem of Algebra). Let \( p \) be a polynomial with \( n := \deg p \geq 1 \). Then \( p \) has \( n \) zeros (counting multiplicity).

Proposition 14.1 (Harmonic Components). Let \( D \subset \mathbb{C} \) be open, and let \( f : D \to \mathbb{C} \) be holomorphic. Then \( \Re f \) and \( \Im f \) are harmonic.

Theorem 14.1 (Harmonic Conjugates). Let \( D \subset \mathbb{C} \) be open and suppose that there exists \((x_0, y_0) \in D\) with the following property: for each \((x, y) \in D\), we have

- \((x, t) \in D\) for each \( t \) between \( y \) and \( y_0 \) and
- \((s, y_0) \in D\) for each \( s \) between \( x \) and \( x_0 \).

Then every harmonic function on \( D \) has a harmonic conjugate.

Corollary 14.1.1. Let \( D \subset \mathbb{C} \) be open, and let \( u : D \to \mathbb{R} \) be harmonic. Then, for each \( z_0 \in D \), there is a neighbourhood \( U \subset D \) of \( z_0 \) such that \( u|_U \) has a harmonic conjugate.

Corollary 14.1.2. Let \( D \subset \mathbb{C} \) be open, and let \( u : D \to \mathbb{R} \) be harmonic. Then \( u \) is infinitely often partially differentiable.

Corollary 14.1.3. Let \( D \subset \mathbb{C} \) be open and connected, and let \( u : D \to \mathbb{R} \) be harmonic. Then the following are equivalent:

(i) \( u \equiv 0 \);

(ii) there exists a nonempty open set \( U \subset D \) with \( u|_U \equiv 0 \).

Corollary 14.1.4. Let \( D \subset \mathbb{C} \) be open, let \( u : D \to \mathbb{R} \) be harmonic, and let \( z_0 \in D \) and \( r > 0 \) be such that \( B_r[z_0] \subset D \). Then we have

\[
u(z_0) = \frac{1}{2\pi} \int_{0}^{2\pi} u(z_0 + re^{i\theta}) \, d\theta.
u(z_0) = \frac{1}{2\pi} \int_{0}^{2\pi} u(z_0 + re^{i\theta}) \, d\theta.
\]
Corollary 14.1.5. Let $D \subset \mathbb{C}$ be open and connected, and let $u: D \to \mathbb{R}$ be harmonic with a local maximum or minimum on $D$. Then $u$ is constant.

Corollary 14.1.6. Let $D \subset \mathbb{C}$ be open, connected, and bounded, and let $u: \overline{D} \to \mathbb{R}$ be continuous such that $u|_D$ is harmonic. Then $u$ attains its maximum and minimum over $\overline{D}$ on $\partial D$.

Theorem 14.2 (Poisson’s Integral Formula). Let $r > 0$, and let $u: B_r(0) \to \mathbb{R}$ be continuous such that $u|_{B_r(0)}$ is harmonic. Then

$$u(z) = \int_0^{2\pi} u(re^{i\theta})P_r(re^{i\theta}, z) \, d\theta$$

holds for all $z \in B_r(0)$.

Theorem 14.3. Let $r > 0$, and let $f: \partial B_r(0) \to \mathbb{R}$ be continuous. Define

$$g: B_r[0] \to \mathbb{C}, \quad z \mapsto \begin{cases} f(z), & z \in \partial B_r(0), \\ \int_0^{2\pi} f(re^{i\theta})P_r(re^{i\theta}, z) \, d\theta, & z \in B_r(0). \end{cases}$$

Then $g$ is continuous and harmonic on $B_r(0)$.

Theorem 14.4. Let $D \subset \mathbb{C}$ be open, and let $f: D \to \mathbb{C}$ have the mean value property such that $|f|$ attains a local maximum at $z_0 \in D$. Then $f$ is constant on a neighbourhood of $z_0$.

Corollary 14.4.1. Let $D \subset \mathbb{C}$ be open, and let $f: D \to \mathbb{R}$ be continuous and have the mean value property, and suppose that $f$ has a local maximum or minimum at $z_0 \in D$. Then $f$ is constant on a neighbourhood of $z_0$.

Corollary 14.4.2. Let $D \subset \mathbb{C}$ be open, connected, and bounded, and let $f: \overline{D} \to \mathbb{R}$ be continuous such that $f|_{D}$ has the mean value property. Then $f$ attains its maximum and minimum on $\partial D$.

Corollary 14.4.3 (Equivalence of Harmonic and Mean-Value Properties). Let $D \subset \mathbb{C}$ be open, and let $f: D \to \mathbb{R}$ be continuous. Then the following are equivalent:

(i) $f$ is harmonic;

(ii) $f$ has the mean value property.
Theorem 17.1 (Conformality at Nondegenerate Points). Let $D_1, D_2 \subset \mathbb{C}$ be open, and let $f: D_1 \to D_2$ be holomorphic. Then $f$ is angle preserving at $z_0 \in D_1$ whenever $f'(z_0) \neq 0$.

Corollary 17.1.1 (Conformality of Biholomorphic Maps). Let $D_1, D_2 \subset \mathbb{C}$ be open and connected, and let $f: D_1 \to D_2$ be biholomorphic. Then $f$ is angle preserving at every point of $D_1$.

Theorem 17.2 (Holomorphic Inverses). Let $D_1, D_2 \subset \mathbb{C}$ be open and connected, and let $f: D_1 \to D_2$ be holomorphic and bijective. Then $f$ is biholomorphic and $\mathcal{Z}(f') = \emptyset$.

Corollary 17.2.1. Let $D \subset \mathbb{C}$ be open and connected, and let $f: D \to \mathbb{C}$ be holomorphic and injective. Then $\mathcal{Z}(f') = \emptyset$.

Theorem 17.3 (Riemann Mapping Theorem). Let $D \subsetneq \mathbb{C}$ be open and connected and admit holomorphic square roots, and let $z_0 \in D$. Then there is a unique biholomorphic function $f: D \to \mathbb{D}$ with $f(z_0) = 0$ and $f'(z) > 0$.

Theorem 17.4 (Simply Connected Domains). The following are equivalent for an open and connected set $D \subset \mathbb{C}$:

(i) $D$ is simply connected;
(ii) $D$ admits holomorphic logarithms;
(iii) $D$ admits holomorphic roots;
(iv) $D$ admits holomorphic square roots;
(v) $D$ is all of $\mathbb{C}$ or biholomorphically equivalent to $\mathbb{D}$;
(vi) every holomorphic function $f: D \to \mathbb{C}$ has an antiderivative;
(vii) $\int_{\gamma} f(\zeta) d\zeta = 0$ for each holomorphic function $f: D \to \mathbb{C}$ and each closed curve $\gamma$ in $D$;
(viii) for every holomorphic function $f: D \to \mathbb{C}$, we have

$$\nu(\gamma, z) f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$$

for each closed curve $\gamma$ in $D$ and all $z \in D \setminus \{\gamma\}$;
(ix) every harmonic function $u: D \to \mathbb{R}$ has a harmonic conjugate.