Review of Math 373

1 Linear Programming

Objective function: Function to be minimized (cost).

Constraints: Linear inequalities or equalities on decision variables.

Feasible solution: Decision variables that satisfy the constraints.

Feasible set: Set of all feasible solutions.

Optimal value: Desired minimum.

Optimal solution: A feasible solution that achieves the optimal cost.

Optimal set: Set of all optimal solutions.

Cost vector: Vector of cost weights for each decision variable.

Linear.

Standard form:

minimize $c^\top x,$
subject to $Ax = b$

$x \geq 0.$

1. Elimination of free variables.

2. Elimination of inequality constraints.
2 Geometry

Polyhedron: 
\[ \{ x \in \mathbb{R}^n : Ax \geq b \} , \]

Polyhedron in standard form: 
\[ \{ x \in \mathbb{R}^n : Ax = b, x \geq 0 \} . \]

Hyperplane: 
\[ \{ x \in \mathbb{R}^n : a^\top x = b \} . \]

Half space: 
\[ \{ x \in \mathbb{R}^n : a^\top x \geq b \} . \]

Convex set.

Convex function.

Convex combination: \( \sum_{i=1}^{k} t_i x_i \), where the non-negative weights \( t_i \) sum up to 1.

Convex hull: the set of all convex combinations.

Absolute values: Minimize \( \sum_{i=1}^{n} c_i |x_i| \), where all \( c_i \geq 0 \):
\[ x_i \to x_i^+ - x_i^- , \quad |x_i| \to x_i^+ + x_i^- . \]

Active constraint.

Extreme point: not an interpolation of two points from polyhedron.

Vertex: lies on separating hyperplane.

Basic feasible solution: all equality constraints active, for a total of \( n \) linearly independent active constraints.

Extreme point=vertex=basic feasible solution.

Adjacent basic solutions.

Edge.

Basic feasible solutions (in standard-form):
\[ i) \ A_{B(1)}, \ldots, A_{B(m)} \text{ are linearly independent;} \]
\[ ii) \text{ if } i \neq B(1), \ldots, B(m), \text{ then } x_i = 0 . \]
\[
B = [A_{B(1)}, \ldots, A_{B(m)}], \quad x_B = \begin{bmatrix}
x_{B(1)} \\
\vdots \\
x_{B(m)}
\end{bmatrix}
\]
\[x_B = B^{-1}b.\]

**Degenerate:** more than \(n\) of the constraints are active at \(x\).

**Degenerate standard-form polyhedra:** some zero basic variables.

**Contain a line.**

**Theorem 2.7:** Suppose that the polyhedron \(P = \{x \in \mathbb{R}^n : a_i^\top x \geq b_i, i = 1, \ldots, m\}\) is nonempty. Then the following are equivalent:

(a) \(P\) has at least one extreme point.

(b) \(P\) does not contain a line.

(c) The set \(\{a_1, \ldots, a_m\}\) contains \(n\) linearly independent vectors.

**Corollary 2.9.1:** Consider the linear programming problem of minimizing \(c^\top x\) over a nonempty polyhedron. Then either the optimal cost is unbounded from below (equals \(-\infty\)) or there exists an optimal solution.

### 3 The Simplex Method

**Basic components of \(j\)th basic direction** (feasible if nondegenerate):

\[
d_B = -B^{-1}A_j.
\]

**Reduced cost:**

\[
c_j = c_j + c_B^\top d_B = c_j - c_B^\top B^{-1}A_j.
\]

**Reduced cost of basic variable:** 0.

**Optimality:** If \(c \geq 0\), \(x\) is optimal. If \(x\) is optimal and non-degenerate, then \(c \geq 0\).