MATH 373 Section A1

Midterm Exam

Dr. J. Bowman

24 October 2019

12:30–13:50

Name (Last, First): ________________________________

Student ID: ____________________________

Email: ____________________________@ualberta.ca

• Scrap paper is supplied.

• No notes or books are permitted.

• All electronic equipment, including calculators, is prohibited. Make certain that cell phones are turned off. Check that you have 5 pages.

• This exam consists of 3 questions, for a total of 27 points.

If anything is unclear, please ask!
1. A linear programming problem in standard form has \( n = 4 \) decision variables \( x_1, x_2, x_3, x_4 \) and \( m = 2 \) equality constraints. The shaded region in the figure represents the projection of the feasible set onto the two-dimensional plane defined by the equality constraints \( Ax = b \).

(a) Identify which of the points \( A, B, C, D \) are basic solutions, which are feasible, which are infeasible, which are non-degenerate, and which are degenerate.

- \( A \): non-degenerate basic infeasible
- \( B \): degenerate basic feasible
- \( C \): non-degenerate basic feasible
- \( D \): non-degenerate basic feasible

(b) Identify all possible pairs of nonbasic variables at \( B \). For each pair identify the basic directions at \( B \) and indicate which are feasible and which are infeasible.

- \( x_1, x_2 \): \( g \) (feasible) and \( f \) (feasible)
- \( x_1, x_3 \): \( h \) (infeasible) and \( f \) (feasible)
- \( x_2, x_3 \): \(-h\) (infeasible) and \( g \) (feasible)
2. Consider the linear programming problem.

\[
\begin{align*}
&\text{minimize} & & x_1 + 2x_2 \\
&\text{subject to} & & x_1 &\leq 3, \\
& & & x_1 + x_2 &\geq 1, \\
& & & x_1, x_2 &\geq 0.
\end{align*}
\]

(a) Put this problem into standard form, minimizing the objective function over the polyhedron \(\{x \in \mathbb{R}^n : Ax = b, x \geq 0\}\).

\[
\begin{align*}
&\text{minimize} & & x_1 + 2x_2 \\
&\text{subject to} & & x_1 + x_3 = 3, \\
& & & x_1 + x_2 - x_4 = 1, \\
& & & x_1, x_2, x_3, x_4 \geq 0.
\end{align*}
\]

(b) Use the procedure we developed in class to find each basic solution to this problem, along with the corresponding basis matrix \(B\). Identify which of these basic solutions are feasible solutions of the standard-form problem; check your work by verifying that these solutions satisfy the constraints. **A graphical solution is not acceptable for any part of this problem.**

\[
A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 1 \end{bmatrix}
\]

Now we consider all possible choices for the basis matrix \(B\), corresponding to all possible ways of choosing two linearly independent column vectors from \(A\).

For \(B(1) = 1\) and \(B(2) = 2\):

\[
B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad x_B = \frac{1}{1} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.
\]

This corresponds to the basic solution \(x = (3, -2, 0, 0)\). Since the second element of \(x\) is negative, we see that this basic solution is infeasible.

For \(B(1) = 1\) and \(B(2) = 3\):

\[
B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad x_B = \frac{1}{1} \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\]

This corresponds to the basic solution \(x = (1, 0, 2, 0)\). Since all elements of \(x\) are non-negative, this is a basic feasible solution.

For \(B(1) = 1\) and \(B(2) = 4\):

\[
B = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad x_B = \frac{1}{1} \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.
\]
This corresponds to the basic solution $x = (3,0,0,2)$. Since all elements of $x$ are non-negative, this is a basic feasible solution.

For $B(1) = 2$ and $B(2) = 3$:

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad x_B = \frac{1}{-1} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}. $$

This corresponds to the basic solution $x = (0,1,3,0)$. Since all elements of $x$ are non-negative, this is a basic feasible solution.

For $B(1) = 2$ and $B(2) = 4$, we see that the corresponding columns of $A$ are linearly dependent.

For $B(1) = 3$ and $B(2) = 4$:

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad x_B = \frac{1}{-1} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}. $$

This corresponds to the basic solution $x = (0,0,3,-1)$. Since the fourth element of $x$ is negative, we see that this basic solution is infeasible.

(c) From your results in part (b), list all of the extreme points $(x_1, x_2)$ of the original problem and their corresponding costs.

The three extreme points $(1,0)$, $(3,0)$ and $(0,1)$ have costs of $1$, $3$, and $2$, respectively.

(d) Identify the optimal extreme point of the original problem and the corresponding optimal cost.

The optimal extreme point is $(1,0)$, with optimal cost $1$.

(e) For the optimal extreme point in the standard-form problem, compute the reduced costs $\hat{c}_j = c_j - c_B^T B^{-1} A_j$ in each basic direction, where $c_j$ is the weight of $x_j$ in the cost function and $c_B$ contains the cost weights of the basic variables. Check that the signs of these reduced costs agree with the optimality theorem we covered in class.

Let $c = (1,2,0,0)$, For the optimal solution $(1,0,2,0)$ we note that $c_B = (1,0)$ and precalculate

$$c_B^T B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{-1} \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}. $$

We find $\hat{c}_2 = 2 - 0 \begin{bmatrix} 0 & 1 \end{bmatrix} = 2 - 1 = 1$ and $\hat{c}_4 = 0 - 0 \begin{bmatrix} 0 & 1 \end{bmatrix} = 0 - (-1) = 1$. We note that all reduced costs are non-negative, consistent with optimality.
3. Let $P$ be a (convex) polyhedron. Complete the proof of the following theorem.

**Theorem 1:** A point $x \in P$ is an extreme point of $P$ if and only if the set $P \setminus \{x\}$ (the set obtained by removing $x$ from $P$) is convex.

Proof: Let $x \in P$. Suppose $P \setminus \{x\}$ is convex. Then it contains every convex combination of points $y, z \in P \setminus \{x\}$. Since $x$ is not in this set, it cannot be expressed as a convex combination of points $y$ and $z$ in $P$. That is, $x$ is an extreme point of $P$.

Suppose $x \in P$ is an extreme point of $P$. If $P \setminus \{x\}$ were not convex, there would exist points $y, z \in P \setminus \{x\} \subset P$ and $t \in (0, 1)$ such that $ty + (1 - t)z \notin P \setminus \{x\}$. But since $P$ is convex, we know that $ty + (1 - t)z \in P$. Thus $ty + (1 - t)z = x$, contradicting the definition of an extreme point. Thus $P \setminus \{x\}$ must be convex.