1. Consider the linear programming problem

\[
\begin{align*}
\text{minimize} & \quad -x_1 + 2x_2 \\
\text{subject to} & \quad x_1 + x_2 \leq 1, \\
& \quad 2x_1 + 3x_2 \leq 2, \\
& \quad x_1 - 2x_2 \geq -2, \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]

(a) Without using the simplex method, show that the problem must have an optimal solution.

It is easy to find a feasible solution, such as \( \mathbf{x} = (0, 0) \). The optimal cost is seen to be bounded since the first inequality implies that \( x_1 \leq 1 \), so that \(-x_1 + 2x_2 \geq -1\). Since the problem is neither infeasible nor unbounded, it must have an optimal solution.

(b) Put this problem into standard form.

For convenience, we first multiply the third constraint by \(-1\) and then put the problem into standard form:

\[
\begin{align*}
\text{minimize} & \quad -x_1 + 2x_2 \\
\text{subject to} & \quad x_1 + x_2 + x_3 = 1, \\
& \quad 2x_1 + 3x_2 + x_4 = 2, \\
& \quad -x_1 + 2x_2 + x_5 = 2, \\
& \quad x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{align*}
\]

(c) Use the simplex method to find an optimal solution and the optimal value for this problem.

Starting from the obvious basic solution \((0, 0, 1, 2, 2)\), with \( B(1) = 3 \), \( B(2) = 4 \), \( B(3) = 5 \), and the identity basis matrix, we perform one iteration of the simplex method:

\[
\begin{array}{c|ccccc}
 & x_1^* & x_2 & x_3 & x_4 & x_5 \\
 \hline
 x_3 & 1 & -1 & 2 & 0 & 0 \\
 x_4 & 2 & 1 & 1 & 1 & 0 \\
 x_5 & 2 & -1 & 2 & 0 & 1 \\
\end{array}
\]

\[
\begin{array}{c|ccccc}
 & x_1 & x_2 & x_3 & x_4 & x_5 \\
 \hline
 1 & 0 & 3 & 1 & 0 & 0 \\
 x_1 & 1 & 0 & 1 & 1 & 0 \\
 x_4 & 0 & 0 & 1 & -2 & 1 \\
 x_5 & 3 & 0 & 3 & 1 & 0 \\
\end{array}
\]
An optimal solution is thus \((1, 0, 0, 0, 3)\), which corresponds to the solution \(x = (1, 0)\) of the original problem, with optimal cost \(-1\).

(d) Is your optimal solution degenerate? Justify your answer.

The optimal solution is degenerate since the basic variable \(x_4\) is 0. Alternatively, in the original problem, we see that there are two variables but three active constraints (including \(x_2 = 0\)).

(e) Verify that the complementary slackness conditions hold for your optimal solution.

The dual problem is

\[
\begin{align*}
\text{minimize} & \quad p_1 + 2p_2 - 2p_3 \\
\text{subject to} & \quad p_1 + 2p_2 + p_3 \leq -1, \\
& \quad p_1 + 3p_2 - 2p_3 \leq 2, \\
& \quad p_1 \leq 0, \quad p_2 \leq 0, \quad p_3 \geq 0.
\end{align*}
\]

Since the third primal constraint isn’t active, we require \(p_3 = 0\). Also, since \(x_1 \neq 0\), the first dual constraint must be active:

\[p_1 + 2p_2 = -1.\]

For example, we could take \(p_1 = -1\) and \(p_2 = 0\). Then \(x = (1, 0)\) and \(p = (-1, 0, 0)\) are feasible solutions to the primal and dual problems, respectively (with optimal cost \(-1\)), that satisfy the complementary slackness conditions. They are therefore both optimal solutions. Note that the dual problem does not have a unique optimal solution since Question 3 does not apply, in view of part (d).

2. The number of servers required at a 24-hour truck stop during each four-hour time interval is given in the following table:

<table>
<thead>
<tr>
<th>Time Interval</th>
<th>Number of Servers</th>
</tr>
</thead>
<tbody>
<tr>
<td>03:00-07:00</td>
<td>2</td>
</tr>
<tr>
<td>07:00-11:00</td>
<td>10</td>
</tr>
<tr>
<td>11:00-15:00</td>
<td>14</td>
</tr>
<tr>
<td>15:00-19:00</td>
<td>8</td>
</tr>
<tr>
<td>19:00-23:00</td>
<td>10</td>
</tr>
<tr>
<td>23:00-03:00</td>
<td>3</td>
</tr>
</tbody>
</table>

The servers report for duty at the beginning of one of these time intervals and their shifts last eight hours (two time intervals). Let \(x_i\) be the number of servers who start their shift at the beginning of the \(i\)th time interval, where \(i = 1, \ldots, 6\). Use the complementary slackness theorem to prove that \(x = (0, 14, 0, 8, 2, 2)\) is a feasible solution that minimizes the total number of servers required each day.
The linear programming problem is

\[
\begin{align*}
\text{minimize} & \quad x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\
\text{subject to} & \quad x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 2, \\
& \quad x_2 + x_3 + x_4 \geq 10, \\
& \quad x_3 + x_4 \geq 14, \\
& \quad x_4 + x_5 \geq 8, \\
& \quad x_5 + x_6 \geq 10, \\
& \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.
\end{align*}
\]

The dual problem is

\[
\begin{align*}
\text{maximize} & \quad 2p_1 + 10p_2 + 14p_3 + 8p_4 + 10p_5 + 3p_6 \\
\text{subject to} & \quad p_1 + p_2 \leq 1, \\
& \quad p_2 + p_3 \leq 1, \\
& \quad p_3 + p_4 \leq 1, \\
& \quad p_4 + p_5 \leq 1, \\
& \quad p_5 + p_6 \leq 1, \\
& \quad p_1, p_2, p_3, p_4, p_5, p_6 \geq 0.
\end{align*}
\]

For the primary problem, we see that all constraints are satisfied but the second and sixth constraints are not active. Complementary slackness then requires that \( p_2 = p_6 = 0 \). For the dual problem, we see that the second, fourth, fifth, and sixth constraint must be active (because the corresponding \( x_i \) is nonzero). Thus

\[
p_3 = 1, \quad p_4 + p_5 = 1, \quad p_5 = 1, \quad p_1 = 1.
\]

We deduce that \( p_4 = 0 \) and observe that the dual solution \( \text{p} = (1, 0, 1, 0, 1, 0) \) also satisfies the remaining first and third constraints and so is feasible. Since the complementary slackness conditions are satisfied, we know that both the primal and dual solutions are optimal.

3. If a linear programming problem in standard form has a non-degenerate basic feasible solution that is optimal, prove that the dual problem has a unique optimal solution. Hint: consider complementary slackness.

Let \( x^* \) be a non-degenerate basic optimal solution to the primal problem. Since the primal problem has an optimal solution, the dual has an optimal solution \( p \). Let \( B(1), \ldots, B(m) \) be a set of basic indices corresponding to \( x^* \) and consider the complementary slackness condition

\[
(c_j - p^T A_j)x^*_j = 0, \quad j = B(1), \ldots, B(m).
\]

Since \( x^* \) is nondegenerate, we know that \( x^*_j > 0 \) for each basic variable \( x_j \), so that

\[
c_j = p^T A_j, \quad j = B(1), \ldots, B(m).
\]
The is just the system of equations
\[ c_B^T = p^T B, \]
which has a unique solution \( p^T = c_B^T B^{-1} \) since the basis \( B = \{ A_j : j = B(1), \ldots, B(m) \} \) is invertible.