1. It is midnight Sunday. Anil has 3 upcoming final exams: Mathematics on Monday evening, Technology on Tuesday evening, and Writing on Wednesday evening. Unfortunately, he has procrastinated too much during the term and now has insufficient time to study thoroughly for all of his exams. Anil must prioritize his study schedule for Monday, Tuesday, and Wednesday. Since he needs 8 hours of sleep, 1 hour for eating, and 3 hours to write each exam, that leaves at most 12 hours per day for studying. Emphasizing the drawbacks of last-minute cramming, the Faculty of Science Advisor has told Anil that in his situation, historical analysis predicts an overall average percentage on the three exams of $4x_1 + 3x_2 + x_3 + 2x_4 + 2x_5 + x_6$, if Anil spends $x_1$ hours studying Mathematics on Monday, $x_2$ and $x_3$ hours studying Technology on Monday and Tuesday, respectively, and $x_4$, $x_5$, and $x_6$ hours studying Writing on Monday, Tuesday, and Wednesday, respectively. However, studying more than 10 hours of Mathematics, more than 8 hours of Technology, or more than 18 hours of Writing will not improve his grade. What is the maximum percentage grade that Anil can expect and how can he allocate his study time to achieve this?

The linear programming problem is

\[
\begin{array}{ll}
\text{minimize} & -4x_1 - 3x_2 - x_3 - 2x_4 - 2x_5 - x_6 \\
\text{subject to} & x_1 \leq 10, \\
& x_2 + x_3 \leq 8, \\
& x_4 + x_5 + x_6 \leq 18, \\
& x_1 + x_2 + x_4 \leq 12, \\
& x_3 + x_5 \leq 12, \\
& x_6 \leq 12, \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.
\end{array}
\]

On introducing slack variables $x_7, \ldots, x_{12}$, we put the problem into standard form and proceed directly to Phase II:

\[
\begin{array}{cccccccccccc}
 & x_1^* & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} \\
0 & -4 & -3 & -1 & -2 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_7^+=10 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_8=8 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
x_9=18 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
x_{10}=12 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
x_{11}=12 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
x_{12}=12 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
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An optimal solution is $x = (10, 2, 6, 0, 6, 12)$, corresponding to an expected grade of 76%. That is, Anil should study 10 hours of Mathematics and 2 hours of Technology on Monday, 6 hours of Technology and 6 hours of Writing on Tuesday, and 12 hours of Writing on Wednesday.

2. A breakfast cereal is required to contain at least 10% protein and not more than 70% carbohydrate by weight. The manufacturer wants to blend four grains $G_1$, $G_2$, $G_3$, and $G_4$ to achieve these requirements. The nutrient content of the grains per weight is given in the following chart:

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<td>% carbohydrate</td>
<td>80</td>
<td>80</td>
<td>70</td>
<td>60</td>
</tr>
<tr>
<td>% protein</td>
<td>5</td>
<td>15</td>
<td>20</td>
<td>25</td>
</tr>
<tr>
<td>cost (cents per kg)</td>
<td>40</td>
<td>80</td>
<td>120</td>
<td>160</td>
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(a) Use the simplex method to determine what weight of each grain the manufacturer should use in order to minimize the overall cost to produce 1kg of cereal. What is the minimal cost? Use units of cents and kilograms.

Hint: to find a basic feasible solution, first use the equality constraint to express the weight of grain $G_1$ in terms of the others weights. This allows you to eliminate one variable (the weight of grain $G_4$ in one kilogram of the cereal) and also one constraint, greatly simplifying the problem!

Let $x_i$ be the weight of grain $G_i$ in 1 kg of cereal.

The linear programming problem is

\[
\begin{align*}
\text{minimize} & \quad 40x_1 + 80x_2 + 120x_3 + 160x_4 \\
\text{subject to} & \quad 80x_1 + 80x_2 + 70x_3 + 60x_4 \leq 70, \\
& \quad 5x_1 + 15x_2 + 20x_3 + 25x_4 \geq 10, \\
& \quad x_1 + x_2 + x_3 + x_4 = 1, \\
& \quad x_1, x_2, x_3, x_4 \geq 0.
\end{align*}
\]

We first use the equality constraint to eliminate $x_4 = 1 - x_1 - x_2 - x_3$. For example, the cost can be expressed as $(40 - 160)x_1 + (80 - 160)x_2 + (120 - 160)x_3 + 160$. To
use the simplex tableau, we will subtract the fixed cost 160 from this amount, to be added back in at the end:

\[
\begin{array}{l}
\text{minimize} \quad -120x_1 - 80x_2 - 40x_3 \\
\text{subject to} \quad 20x_1 + 20x_2 + 10x_3 \leq 10, \\
-20x_1 - 10x_2 - 5x_3 \geq -15, \\
x_1, x_2, x_3 \geq 0.
\end{array}
\]

Next we put the problem into standard form, introducing slack variables \( x_5 \) and \( x_6 \):

\[
\begin{array}{l}
\text{minimize} \quad -120x_1 - 80x_2 - 40x_3 \\
\text{subject to} \quad 20x_1 + 20x_2 + 10x_3 + x_5 = 10, \\
20x_1 + 10x_2 + 5x_3 + x_6 = 15, \\
x_1, x_2, x_3, x_5, x_6 \geq 0.
\end{array}
\]

We notice that the basic indices \( B(1) = 5, B(2) = 6 \) give us a basic feasible solution with identity basis matrix:

\[
\begin{array}{c|ccccc}
& x^*_1 & x_2 & x_3 & x_5 & x_6 \\
\hline
x^\dagger_5 & 0 & -120 & -80 & -40 & 0 \\
x^\dagger_6 & 10 & 20 & 20 & 10 & 1 \\
\end{array}
\]

One iteration of the simplex method then bring us immediately to the optimal solution:

\[
\begin{array}{c|ccccc}
& x_1 & x_2 & x_3 & x_5 & x_6 \\
\hline
x_1 & 60 & 0 & 40 & 20 & 6 \\
x_6 & 1/2 & 1 & 1/2 & 1/20 & 0 \\
\end{array}
\]

Recalling that \( x_4 = 1 - x_1 - x_2 - x_3 \), the optimal solution is \( x = (1/2, 0, 0, 1/2, 0, 5) \). If we now add back in the fixed cost 160 to the optimal cost -60 from the tableau, we deduce the actual optimal cost to be 100 cents. Thus, the optimal recipe is a mixture of 1/2 kg of \( G_1 \) and 1/2 kg of \( G_4 \), which will cost $1 per kg.

(b) By how much would the price of grain \( G_3 \) have to drop for the blend you found in part (a) to be no longer optimal? Do not do any further simplex iterations. Hint: Make use of the formula for \( \bar{c} \). Does \( c_B \) change?

Suppose that \( c_3 \) changes to \( c_3 - \delta \). Optimality requires that \( \bar{c} \geq 0 \). Since \( x_3 \) is a nonbasic variable, \( c_B \) does not change and hence the only reduced cost that will change is \( c_3 \). To violate optimality, we require that

\[
c_3 - \delta - c^\dagger_B B^{-1} A_3 = \bar{c}_3 - \delta < 0.
\]

That is, the blend given in part(a) would no longer be optimal once the price of \( G_3 \) drops by more than \( \bar{c}_3 = 20 \) cents. At that point the requirements could be achieved for lower cost by making the cereal out of \( G_3 \) only.
3. Consider a two-player game $G$ with the following payoff matrix pair

$$A = \begin{pmatrix} 3 & 0 \\ 2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 \\ 5 & 5 \end{pmatrix}. \quad (1)$$

(a) Denote the strategy of the first player by $y$ and of the second player by $z$. Analytically determine the set of Nash equilibria of the game. Write down the corresponding linear programming problems.

We have

$$y = \begin{bmatrix} y_1 \\ 1 - y_1 \end{bmatrix}, \quad y_1 \geq 0, \quad z = \begin{bmatrix} z_1 \\ 1 - z_1 \end{bmatrix}, \quad z_1 \geq 0.$$  

A Nash equilibrium $(y, z)$ must satisfy

$$\forall x, \quad u_1(y, z) \geq u_1(x, z), \quad (2)$$

$$\forall x, \quad u_2(y, z) \geq u_2(y, x) \quad (3)$$

Condition (2) implies that $y$ must maximize $u_1(x, z)$ over $x$, and condition (3) implies that $z$ must maximize $u_2(y, x)$ over $x$. We find

$$y = \arg\max_{x \in \Delta_2} x^T A z = \arg\max_{x_1} \left[ x_1 \cdot 1 - x_1 \right] \begin{bmatrix} 3 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ 1 - z_1 \end{bmatrix} = \arg\max_{x_1} \left[ x_1 \cdot 1 - x_1 \right] \begin{bmatrix} 3 z_1 \\ 2 \end{bmatrix}.$$ 

So based on the ordering of $3z_1$ and $2$, $y_1$ is given by

$$\begin{cases} y_1 = 1 & z_1 > \frac{2}{3} \\ y_1 \in [0, 1] & z_1 = \frac{2}{3} \\ y_1 = 0 & z_1 < \frac{2}{3} \end{cases}. \quad (4)$$

Likewise,

$$z = \arg\max_{x_1} y^T B x = \arg\max_{x_1} \left[ y_1 \cdot 1 - y_1 \right] \begin{bmatrix} 3 & 0 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ 1 - x_1 \end{bmatrix} = \arg\max_{x_1} \left[ 5 - 2y_1 \cdot 5 - 5y_1 \right] \begin{bmatrix} x_1 \\ 1 - x_1 \end{bmatrix}.$$ 

So based on the ordering of $5 - 2y_1$ and $5 - 5y_1$, $z_1$ is given by

$$\begin{cases} z_1 = 1 & 5 - 2y_1 > 5 - 5y_1 \\ z_1 \in [0, 1] & 5 - 2y_1 = 5 - 5y_1 \Rightarrow \begin{cases} z_1 = 1 & y_1 > 0 \\ z_1 \in [0, 1] & y_1 = 0 \end{cases} \\ z_1 = 0 & 5 - 2y_1 < 5 - 5y_1 \end{cases}. \quad (5)$$

Now (4) and (5) result in the following possibilities

$$\begin{cases} y_1 = 1, z_1 = 1 \\ z_1 = \frac{2}{3}, y_1 = 0 \Rightarrow \begin{cases} y_1 = 1, z_1 = 1 \\ y_1 = 0, z_1 \leq \frac{2}{3} \Rightarrow \end{cases} \end{cases} \quad \begin{cases} y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, z = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, z = \begin{bmatrix} \frac{z_1}{1 - z_1} \\ 1 - \frac{z_1}{1 - z_1} \end{bmatrix}, 0 \leq z_1 \leq \frac{2}{3} \end{cases}. \quad (6)$$
The corresponding linear programming problems are, given \((z_1, z_2)\):

\[
\begin{align*}
\text{maximize} & \quad 3z_1 x_1 + 2x_2 \\
\text{subject to} & \quad x_1 + x_2 = 1, \\
& \quad x_1, x_2 \geq 0,
\end{align*}
\]

and, given \((y_1, y_2)\):

\[
\begin{align*}
\text{maximize} & \quad (5 - 2y_1)x_1 + 5(1 - y_1)x_2 \\
\text{subject to} & \quad x_1 + x_2 = 1, \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]

(b) Consider another two-player game \(G'\) with payoff matrices

\[
A' = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \quad B' = \begin{pmatrix} 6 & 0 \\ 10 & 10 \end{pmatrix}.
\]

Using part (a), is it possible to make a statement about the set of Nash equilibria of \(G'\) by just comparing the payoff matrices in Eqs. (1) and (6)? Explain. Hint: Do the Nash equilibria change if you add a constant to all entries of a column of \(A\)?

Adding a constant to all entries of any column of \(A\) does not change the set of Nash equilibria:

\[
\arg \max_{x \in \Delta_2} \left( \sum_{i,j} x_i (A_{ij} + c_j) z_j \right) = \arg \max_{x \in \Delta_2} \left( \sum_{i,j} x_i A_{ij} z_j + \sum_i x_i \sum_j c_j z_j \right)
\]

\[
= \arg \max_{x \in \Delta_2} \left( \sum_{i,j} x_i A_{ij} z_j + \sum_j c_j z_j \right)
\]

\[
= \arg \max_{x \in \Delta_2} \sum_{i,j} x_i A_{ij} z_j.
\]

The same holds true when adding a constant to all entries of any row of \(B\) or when a payoff matrix is multiplied by a constant.

Now if we add \(-3\) to the first column of \(A\) and \(-1\) to the second column of \(A\), we get \(A'\). Moreover, if we multiply \(B\) by 2, we get \(B'\). Hence, \(G\) and \(G'\) have the same Nash equilibria.