1. Staples has two warehouses $W_1$ and $W_2$ with 10 and 3 tablets in stock, respectively, and wishes to restock two stores $S_1$ and $S_2$ with 6 and 7 tablets, respectively. If the cost to ship an item from $W_i$ to $S_j$ is $c_{ij}$, where $(c_{11}, c_{12}, c_{21}, c_{22}) = (1, 3, 2, 3)$ (in dollars), use the two-phase simplex method to determine the number of tablets $x_{ij}$ Staples should ship from warehouse $i$ to store $j$ so that the overall transportation cost is minimized.

It is convenient to relabel $(x_{11}, x_{12}, x_{21}, x_{22})$ as $(x_1, x_2, x_3, x_4)$. The linear programming problem is

\[
\begin{align*}
\text{minimize} & \quad x_1 + 3x_2 + 2x_3 + 3x_4 \\
\text{subject to} & \quad x_1 + x_2 = 10, \\
& \quad x_3 + x_4 = 3, \\
& \quad x_1 + x_3 = 6, \\
& \quad x_2 + x_4 = 7, \\
& \quad x_1, x_2, x_3, x_4 \geq 0.
\end{align*}
\]

Using the two-phase simplex method, the problem can be solved as follows.

**Phase I:**

**Step 1.** Multiply every constraint with a negative right-hand side by $-1$, so that $b \geq 0$. In this case, the constraints are already in this form.

**Step 2.** In order to find a basic feasible solution, the artificial variables $x_5, \ldots, x_8$ are introduced to form the auxiliary problem

\[
\begin{align*}
\text{minimize} & \quad x_5 + x_6 + x_7 + x_8 \\
\text{subject to} & \quad x_1 + x_2 + x_5 = 10, \\
& \quad x_3 + x_4 + x_6 = 3, \\
& \quad x_1 + x_3 + x_7 = 6, \\
& \quad x_2 + x_4 + x_8 = 7, \\
& \quad x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0.
\end{align*}
\]

Using the two-phase simplex method, the problem can be solved as follows.

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<tr>
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Phase II:

\[
\begin{array}{cccccccc}
   & x_1 & x_2 & x_3 & x_4^* & x_4 & x_5 & x_6 & x_7 & x_8 \\
-30 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
   & x_1 & x_2 & x_3 & x_4^* & x_4 & x_5 & x_6 & x_7 & x_8 \\
 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
 x_2=7 & 0 & 1 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \\
x_3=3 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
x_1=3 & 1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\
x_8=0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 \\
\end{array}
\]

The minimal transportation cost of $27 is realized by the optimal solution $(6, 4, 0, 3)$, where Warehouse 1 ships 6 tablets to Store 1 and 4 tablets to Store 2 and Warehouse 2 ships 3 tablets to Store 2.

2. Use the two-phase simplex method to determine the optimal solution and optimal cost of the linear programming problem. You will need to put the problem into standard form, labelling the decision variables with sequential integer indices.

\[
\begin{array}{cccc}
   & x_1 & x_2 & x_3 & x_4 \\
-27 & 0 & 0 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
   & x_1 & x_2 & x_3 & x_4 \\
 x_2=4 & 0 & 1 & -1 & 0 \\
x_4=3 & 0 & 0 & 1 & 0 \\
x_1=6 & 1 & 0 & 1 & 0 \\
\end{array}
\]

minimize \( x_1 + 3x_2 \)
subject to \( x_1 - 2x_2 \geq 4, \)
\( x_1 - x_2 = 5, \)
\( x_1 + 2x_2 \geq -6, \)
\( x_1, x_2 \geq 0. \)
Using the two-phase simplex method, the problem can be solved as follows.

**Phase I:**

**Step 1.** Multiply every constraint with a negative right-hand side by $-1$, so that $b \geq 0$, and add the slack variables $x_3$ and $x_4$:

\[
\begin{align*}
\text{minimize} & \quad x_1 + 3x_2 \\
\text{subject to} & \quad x_1 - 2x_2 - x_3 = 4, \\
& \quad x_1 - x_2 = 5, \\
& \quad -x_1 - 2x_2 + x_4 = 6, \\
& \quad x_1, x_2, x_3, x_4 \geq 0.
\end{align*}
\]

**Step 2.** In order to find a basic feasible solution, the artificial variables $x_5$ and $x_6$ are introduced to form the auxiliary problem

\[
\begin{align*}
\text{minimize} & \quad x_5 + x_6 \\
\text{subject to} & \quad x_1 - 2x_2 - x_3 + x_5 = 4, \\
& \quad x_1 - x_2 + x_6 = 5, \\
& \quad -x_1 - 2x_2 + x_4 = 6, \\
& \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.
\end{align*}
\]

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Phase II:

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<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

Thus the optimal solution is $(5,0,1,11)$, which corresponds to the optimal solution $(5,0)$ to the original problem, with optimal cost 5.

3. Consider the linear programming problem

$$\text{minimize } x_1 + 2x_2$$
$$\text{subject to } x_1 + 3x_2 \leq 3,$$
$$2x_1 + x_2 = 1,$$
$$x_1 \geq 0.$$  

(a) Put this problem into standard form, labelling the new decision variables with sequential integer indices.

$$\text{minimize } x_1 + 2x_2 - 2x_3$$
$$\text{subject to } x_1 + 3x_2 - 3x_3 + x_4 = 3,$$
$$2x_1 + x_2 - x_3 + x_5 = 1,$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0.$$  

(b) Use the two-phase simplex method to show that the optimal cost of this problem is $-\infty$.

Phase I:

Step 1. Multiply every constraint with a negative right-hand side by $-1$, so that $b \geq 0$. In this case, the constraints are already in this form.

Step 2. In order to find a basic feasible solution, the artificial variable $x_5$ is introduced to form the auxiliary problem

$$\text{minimize } x_1 + 3x_2 - 3x_3 + x_4 + x_5$$
$$\text{subject to } x_1 + 3x_2 - 3x_3 + x_4 = 3,$$
$$2x_1 + x_2 - x_3 + x_5 = 1,$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0.$$
Since all entries in column $x_3$ are negative, the simplex method terminates with optimal cost $-\infty$.

(c) For each $t \geq 0$, use the final simplex iteration to determine a feasible solution $x(t)$ such that $\lim_{t \to \infty} c^\top x(t) = -\infty$. Prove your assertion by demonstrating the limit.

Let $x_0 = (1/2, 0, 0, 5/2)$ be the final vertex encountered in Phase II. For the third basic direction, we deduce from $d_B = -B^{-1} A_3 = (5/2, 1/2)$ that $d = (1/2, 0, 1, 5/2)$. Thus

$$x(t) = x_0 + td = \left( \frac{1}{2} + \frac{1}{2} t, 0, t, \frac{5}{2} + \frac{5}{2} t \right).$$
Using the cost vector \( c(t) = (1, 2, -2, 0) \) from the standard-form problem, we see that the optimal cost is

\[
\lim_{t \to \infty} c^t x(t) = \lim_{t \to \infty} \frac{1}{2} + \frac{1}{2} t - 2t = \lim_{t \to \infty} \frac{1}{2} - \frac{3}{2} t = -\infty.
\]