1. Use complementary slackness to determine the set \( \{ c_1, c_2, c_3 \} \) for which \( (1, 1, 0) \) is an optimal solution to the linear programming problem

\[
\begin{align*}
\text{minimize} & \quad c_1 x_1 + c_2 x_2 + c_3 x_3 \\
\text{subject to} & \quad x_1 - x_2 + x_3 \leq 0, \\
& \quad 2x_1 + x_2 - x_3 = 3, \\
& \quad x_1 - x_2 - x_3 \leq 1, \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}
\]

The dual problem is

\[
\begin{align*}
\text{maximize} & \quad 3p_2 + p_3 \\
\text{subject to} & \quad p_1 + 2p_2 + p_3 \leq c_1, \\
& \quad -p_1 + p_2 - p_3 \leq c_2, \\
& \quad p_1 - p_2 - p_3 \leq c_3, \\
& \quad p_1 \leq 0, p_3 \leq 0.
\end{align*}
\]

Since only the first two constraints of the primal problem are active at \( \mathbf{x} = (1, 1, 0) \), we require \( p_3 = 0 \). Moreover, since \( x_1 \) and \( x_2 \) are nonzero, the first two dual constraints must also be active:

\[
\begin{align*}
p_1 + 2p_2 &= c_1, \\
-p_1 + p_2 &= c_2.
\end{align*}
\]

On adding these equations, we find that \( p_2 = (c_1 + c_2)/3 \). We require that

\[
0 < p_1 = p_2 - c_2 = c_1/3 - 2c_2/3,
\]

which implies that \( c_1 \leq 2c_2 \). We also require that \( \mathbf{x} \) and \( \mathbf{p} \) be feasible solutions. The given solution \( (1, 1, 0) \) satisfies all six primal constraints (including positivity). For the dual solution \( (p_1, p_2, 0) \) to satisfy all six dual constraints, we require in addition that \( p_1 - p_2 \leq c_3 \), which reduces to \( -c_2 \leq c_3 \). The complementary slackness theorem then guarantees that the solution set of cost coefficients such that \( (1, 1, 0) \) is an optimal solution to the primal problem is

\[
\{(c_1, c_2, c_3) : c_1 \leq 2c_2, c_3 \geq -c_2\}.
\]
2. Use an appropriate version of Farkas’ lemma to show that the system of equations
\[
\begin{align*}
x_1 + 2x_2 + 3x_3 & \leq 1, \\
5x_1 + x_2 + 6x_3 & \leq -1, \\
x_1, x_2, x_3 & \geq 0.
\end{align*}
\]
is inconsistent.

Consider the linear programming problem
\[
\begin{align*}
\text{maximize} & \quad 0x_1 + 0x_2 + 0x_3, \\
\text{subject to} & \quad x_1 + 2x_2 + 3x_3 \leq 1, \\
& \quad 5x_1 + x_2 + 6x_3 \leq -1, \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}
\]
The dual problem is
\[
\begin{align*}
\text{minimize} & \quad p_1 - p_2 \\
\text{subject to} & \quad p_1 + 5p_2 \geq 0, \\
& \quad 2p_1 + p_2 \geq 0, \\
& \quad 3p_1 + 6p_2 \geq 0, \\
& \quad p_1, p_2 \geq 0.
\end{align*}
\]
If we let \( p_1 = 0 \) and take \( p_2 \) to be arbitrarily large, we see that the constraints of the dual problem are satisfied, but the cost approaches \(-\infty\). That is, the dual problem is unbounded. Corollary 4.3.1 then guarantees that the primary problem is infeasible. That is, the original constraints are inconsistent.

3. Kourosh is taking a summer course at the university. He wants to spend at least 8 hours of studying for this course. On the other hand, he doesn’t like to be lonely and wants to spend at least 10 hours with his friends. However, being new to the university, Kourosh has only one friend. His friend is also taking the same course. Sometimes Kourosh studies together with his friend, which counts towards the 10 hours he needs to be with his friend. Kourosh does not want to be overwhelmed with extracurricular activities. So he decides to limit the time he spends with his friend doing extracurricular activities to at most 12 hours. Kourosh also has a summer job and finds it difficult to distribute his time. However, he feels that his problem is a linear programming problem. He decides to ask one of the students of MATH 373 for some advice. He finds you!

Kourosh explains his situation to you and asks “how should I distribute my time so that I learn the topics in my course as much as possible.” He clarifies that for each hour he spends on the course, either alone or with his friend, he gains one “unit of learning.”

(a) Formulate his problem as a linear programming problem, using the following notation:
let $x_1$ represent the time Kourash spends studying for his course alone;
let $x_2$ represent the time Kourash spends studying for his course with his friend;
let $x_3$ represent the time Kourash spends on extracurricular activities with his friend.

$$\begin{align*}
\text{maximize} & \quad x_1 + x_2 \\
\text{subject to} & \quad x_1 + x_2 \geq 8, \\
& \quad x_2 + x_3 \geq 10, \\
& \quad x_3 \leq 12, \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}$$

(b) Show that the problem is unbounded.
When $x_1 = x_3 = 0$ and $x_2 \to \infty$, all of the constraints are satisfied and the objective function approaches $\infty$.

(c) Kouros does not understand what you mean by “the problem is unbounded.” Interpret this statement in terms of his situation.
It indicates that he could spend an unlimited amount of time on his course, which is unrealistic. He is obviously missing a constraint.

(d) After your explanation, Kouros recalls that he has forgotten to tell you that every constraint he mentioned was for just a week. In other words, he wants to spend at least 8 hours on the course per week, 10 hours with his friend per week, and at most 12 hours of extracurricular activities per week. Moreover, he wants to maximize his ‘learning’ per week. Without solving the new problem, show that it now has an optimal solution.

$$\begin{align*}
\text{maximize} & \quad x_1 + x_2 \\
\text{subject to} & \quad x_1 + x_2 \geq 8, \\
& \quad x_2 + x_3 \geq 10, \\
& \quad x_3 \leq 12, \\
& \quad x_1 + x_2 + x_3 \leq 168, \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}$$

We know that a linear programming problem is either infeasible, unbounded, or has an optimal solution. There are many obvious feasible solutions, such as $x_1 = x_3 = 0$ and $x_2 = 10$. The cost is not unbounded since since $x_1 + x_2 \leq 168 - x_3 \leq 168$ since $x_3 \geq 0$. The only remaining possibility is that the problem has an optimal solution.

(e) Kouros then tells you that he has changed his mind: instead of maximizing his ‘learning’, he wants to minimize the time he spends on his course, either alone or with his friend. Write the new problem as a linear programming problem.
(f) Show that the “per week” constraint is no longer required (in other words, if there is an optimal solution $\mathbf{x}^*$ to the minimization problem without the “per week” constraint, then $\mathbf{x}^*$ is also an optimal solution with the “per week” constraint).

If there were an optimal solution that violated the “per week” constraint then since $x_3 \leq 12$ we see that

$$x_1 + x_2 > 168 - x_3 \geq 168 - 12 = 156.$$  

But the optimal cost $x_1 + x_2$ is clearly less than this since the optimal cost for the feasible solution $(0, 10, 0)$ is 10, which is less than 156. So the presence of the “per week” constraint does not affect the optimal solution to the minimization problem.

**Ignore the “per week” constraint for the remaining parts.**

(g) Before you start solving the minimization problem, Kourosh says “I think the best I can do is to study only with my friend for 8 hours and spend 2 hours with my friend on extracurricular activities.” Use complementary slackness to check whether Kourosh’s answer is correct.

The dual problem is

$$\text{maximize } 8p_1 + 10p_2 + 12p_3$$

subject to

$$p_1 \leq 1,$$

$$p_1 + p_2 \leq 1,$$

$$p_2 + p_3 \leq 0,$$

$$p_1 \geq 0, p_2 \geq 0, p_3 \leq 0.$$  

The feasible solution $\mathbf{x} = [0, 8, 2]$ makes the first two primal constraints active, but not the third, so $p_3 = 0$. Since $x_2$ and $x_3$ are nonzero, the corresponding two dual constraints must be active:

$$p_1 + p_2 = 1,$$

$$p_2 + p_3 = 0,$$

so $p_3 = -p_2$ and $p_1 = 1 - p_2$. For a dual feasible solution, the sign conditions on the prices requires that $0 \leq p_2 \leq 1$. For example, we could take $p = (1, 0, 0)$ as a feasible dual solution. By complementary slackness, the solution $\mathbf{x} = [0, 8, 2]$ is an optimal solution of the primal problem.
(h) After your above answer, Kourosh starts wondering what you have actually learned in your MATH 373 course, because he already found the optimal solution himself. He then tells you “I am not happy with this solution though, because I prefer some part of the time that I spend on my course to be without my friend.” You then say “OK, the problem may have more than one optimal solution. Let me see.” Write the problem, without the “per week,” constraint in standard form, labelling any new decision variables with sequential integer indices.

\[
\begin{align*}
\text{minimize} & \quad x_1 + x_2 \\
\text{subject to} & \quad x_1 + x_2 - x_4 = 8, \\
& \quad x_2 + x_3 - x_5 = 10, \\
& \quad x_3 + x_6 = 12, \\
& \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.
\end{align*}
\]

(i) Find a basis that leads to an infeasible basic solution (try to make the basis as simple as possible). Provide both the basis and the infeasible basic solution.

Let us choose \( x_4, x_5, \) and \( x_6 \) as our basic variables. If we multiply the first two constraints by \(-1:\)

\[
\begin{align*}
\text{minimize} & \quad x_1 + x_2 \\
\text{subject to} & \quad -x_1 - x_2 + x_4 = -8, \\
& \quad -x_2 - x_3 + x_5 = -10, \\
& \quad x_3 + x_6 = 12, \\
& \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0,
\end{align*}
\]

then the basis matrix is just the \(3 \times 3\) identity matrix. An infeasible basic solution is \((0, 0, 0, -8, -10, -12)\).

(j) Use this basis to construct the first tableau and iterate until you find the optimal solution and cost. Use Bland’s rule for deciding on the variables that exit and enter the basis; namely, between those variables that are eligible to exit, choose the one with the smallest index, and also between those variables that are tied for entering, choose the one with the smallest index.

Since \(c_B = (0, 0, 0)\), the reduced cost vector is equal to the cost vector \(c\). Since \(c \geq 0\), we can apply the dual simplex method:

\[
\begin{array}{ccccccc|cc}
 & x_1^* & x_2 & x_3 & x_4 & x_5 & x_6 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
-8 & -1 & -1 & 0 & 1 & 0 & 0 \\
-10 & 0 & -1 & -1 & 0 & 1 & 0 \\
12 & 0 & 0 & 1 & 0 & 0 & 1 \\
\end{array}
\]
The optimal solution is thus \((0, 8, 2, 0, 0, 10)\), with optimal value 8.

(k) Is your optimal solution degenerate? Why or why not?

No, it is not degenerate since all basic variables are nonzero.

(l) Is it unique? Why or why not?

No, because the solution is nondegenerate and the reduced cost of the nonbasic variable \(x_1\) (or \(x_5\)) is zero. This means that the cost will not change if we move in the 1st (or 5th) basic direction.

(m) Does it satisfy Kourosh’s preference of spending some time without his friend on the course?

No, it does not: \(x_1 = 0\).

(n) Kourosh then says that I also think that an alternative optimal solution is to spend 4 hours on my course alone, 4 hours on my course with my friend and 6 hours on extracurricular activities with my friend. Show that his alternative solution is indeed optimal.

The alternative solution \(\mathbf{x} = (4, 4, 6, 0, 0, 6)\) satisfies all of the constraints, so it is feasible, and it also achieves the optimal cost of 8.

(o) Kourosh then says he has heard that people knowing linear optimization use something called “the simplex method.” You reply “yes, let me use it to find a solution for you.” But then Kourosh stops you and says that “I just want to know whether this simplex method can ever find my alternative solution?” Without doing the simplex method, explain why it cannot find Kourosh’s optimal solution. (Recall what kind of solutions the simplex

The optimal solution is thus \((0, 8, 2, 0, 0, 10)\), with optimal value 8.

(k) Is your optimal solution degenerate? Why or why not?

No, it is not degenerate since all basic variables are nonzero.

(l) Is it unique? Why or why not?

No, because the solution is nondegenerate and the reduced cost of the nonbasic variable \(x_1\) (or \(x_5\)) is zero. This means that the cost will not change if we move in the 1st (or 5th) basic direction.

(m) Does it satisfy Kourosh’s preference of spending some time without his friend on the course?

No, it does not: \(x_1 = 0\).

(n) Kourosh then says that I also think that an alternative optimal solution is to spend 4 hours on my course alone, 4 hours on my course with my friend and 6 hours on extracurricular activities with my friend. Show that his alternative solution is indeed optimal.

The alternative solution \(\mathbf{x} = (4, 4, 6, 0, 0, 6)\) satisfies all of the constraints, so it is feasible, and it also achieves the optimal cost of 8.

(o) Kourosh then says he has heard that people knowing linear optimization use something called “the simplex method.” You reply “yes, let me use it to find a solution for you.” But then Kourosh stops you and says that “I just want to know whether this simplex method can ever find my alternative solution?” Without doing the simplex method, explain why it cannot find Kourosh’s optimal solution. (Recall what kind of solutions the simplex
method provides in each iteration, and find what is special about Kourosh’s solution.)

Kourosh’s solution is not a vertex since it has 2 just two zeros whereas every vertex of the polyhedron in part (h) must have \(6 - 3 = 3\) nonbasic (zero) variables. The simplex method only searches the vertices of the polyhedron, so it never finds Kourosh’s solution.

(p) When you provide a negative answer to Kourosh, he gets more puzzled of whether what you have learned during MATH 373 is useful at all and tells you “I don’t think I will take the MATH 373 course because my current knowledge seems to be more useful than what I can learn from that course.” You become so sad and almost want to cry. Kourosh leaves but returns a month later and says that “I think that I should spend less time studying by myself. So I want the time I spend on my course to be weighted more in the cost function. However, I am not sure how much more I want to weight it.” You reply “No problem, I will give you the general solution.” Use parametric programming to solve Kourosh’s problem. Use the information from the first tableau in part (j) to construct your first tableau.

Let us weight \(x_1\) by \(t\geq 1\) in the cost function:

\[
\begin{align*}
\text{minimize} & \quad tx_1 + x_2 \\
\text{subject to} & \quad x_1 + x_2 - x_4 = 8, \\
& \quad x_2 + x_3 - x_5 = 10, \\
& \quad x_3 + x_6 = 12, \\
& \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.
\end{align*}
\]

Using the dual simplex method, we find

\[
\begin{array}{c|ccccccc}
& x_1 & x_2^* & x_3 & x_4 & x_5 & x_6 \\
\hline
x_4^* & 0 & t & 1 & 0 & 0 & 0 & 0 \\
x_5 & -8 & -1 & -1 & 0 & 1 & 0 & 0 \\
x_6 & -10 & 0 & -1 & -1 & 0 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{c|ccccccc}
& x_1 & x_2 & x_3^* & x_4 & x_5 & x_6 \\
\hline
-8 & t - 1 & 0 & 0 & 1 & 0 & 0 \\
x_2 & 8 & 1 & 1 & 0 & -1 & 0 & 0 \\
x_3^* & -2 & 1 & 0 & -1 & -1 & 1 & 0 \\
x_6 & 12 & 0 & 0 & 1 & 0 & 0 & 1 \\
\end{array}
\]

\[
\begin{array}{c|ccccccc}
& x_1 & x_2 & x_3 & x_4 & x_5^* & x_6 \\
\hline
-8 & t - 1 & 0 & 0 & 1 & 0 & 0 \\
x_2 & 8 & 1 & 1 & 0 & -1 & 0 & 0 \\
x_3 & 2 & -1 & 0 & 1 & 1 & -1 & 0 \\
x_6 & 10 & 1 & 0 & 0 & -1 & 1 & 1 \\
\end{array}
\]
Since \( t \geq 1 \), we thus arrive again at the optimal solution \((0, 8, 2, 0, 0, 10)\), with optimal value 8.

(q) After Kourosh sees your answer, he starts to appreciate what you have learned in MATH 373. Kourosh then asks, “What if I want the absolute value of the difference between the time I spend on my course alone and the time I spend on my course together with my friend to be less than or equal to 1?” Write this revised problem in standard form.

\[
\begin{align*}
\text{minimize} & \quad x_1 + x_2 \\
\text{subject to} & \quad -x_1 - x_2 + x_4 = -8, \\
& \quad -x_2 - x_3 + x_5 = -10, \\
& \quad x_3 + x_6 = 12, \\
& \quad x_1 - x_2 + x_7 = 1, \\
& \quad -x_1 + x_2 + x_8 = 1, \\
& \quad x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0.
\end{align*}
\]

**Alternative solution:** Instead, one can write \( x_1 - x_2 = x_7 - x_8 \) and \(|x_1 - x_2| = x_7 + x_8\) where \( x_7 \) and \( x_8 \) are new nonnegative variables. The linear programming problem becomes

\[
\begin{align*}
\text{minimize} & \quad x_1 + x_2 \\
\text{subject to} & \quad -x_1 - x_2 + x_4 = -8, \\
& \quad -x_2 - x_3 + x_5 = -10, \\
& \quad x_3 + x_6 = 12, \\
& \quad -x_1 + x_2 - x_7 + x_8 = 0, \\
& \quad x_7 + x_8 + x_9 = 1, \\
& \quad x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \geq 0.
\end{align*}
\]

However, this approach would require more work for the next question in that there are now two new variables in addition to two new constraints (not counting the slack variables already accounted for in the procedure for adding new inequality constraints).

(r) Use sensitivity analysis to find the optimal solution to the revised problem, by adding the new constraint to the last tableau in part (j).

Let us rewrite the final fourth and fifth constraints in the form

\[
\begin{align*}
-x_1 + x_2 & - x_7 = -1, \\
x_1 - x_2 - x_8 & = -1.
\end{align*}
\]

The solution \((0, 8, 2, 0, 0, 10)\) in part (j) satisfies the first constraint but not the second.

To add the first constraint, let \( a_{m+1}^T = [-1, 1, 0, 0, 0, 0] \). The basic components of \( a_{m+1} \) are \( a^T = [0, 1, 0] \). We precompute

\[
\begin{align*}
a^T B^{-1} A - a_{m+1}^T &= \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}.
\end{align*}
\]
We see that the new slack variable $x_7$ must be 9 in order to satisfy the constraint $-x_1 + x_2 - x_7 = -1$.

So the first tableau is

\[
\begin{array}{cccccccc}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
-8 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
x_3 = & 2 & -1 & 0 & 1 & 1 & -1 & 0 & 0 \\
x_2 = & 8 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\
x_6 = & 10 & 1 & 0 & 0 & -1 & 1 & 1 & 0 \\
x_7 = & 9 & 2 & 0 & 0 & -1 & 0 & 0 & 1 \\
\end{array}
\]

For the second constraint, let $a_{m+1}^\top = [1, -1, 0, 0, 0, 0]$. The basic components of $a_{m+1}$ are $a^\top = [0, -1, 0, 0]$. We precompute

\[
a^\top B^{-1} A - a_{m+1}^\top = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.
\]

We see that the new slack variable $x_8$ must be $-7$ in order to satisfy the constraint $x_1 - x_2 - x_8 = -1$.

So the second tableau is

\[
\begin{array}{cccccccc}
 & x_1^* & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
-8 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
x_3 = & 2 & -1 & 0 & 1 & 1 & -1 & 0 & 0 \\
x_2 = & 8 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\
x_6 = & 10 & 1 & 0 & 0 & -1 & 1 & 1 & 0 \\
x_7 = & 9 & 2 & 0 & 0 & -1 & 0 & 0 & 1 \\
x_8^* = & -7 & -2 & 0 & 0 & 1 & 0 & 0 & 1 \\
\end{array}
\]

After one iteration of the dual simplex method, we find

\[
\begin{array}{cccccccc}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
-8 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
x_3 = & 11/2 & 0 & 0 & 1 & 1/2 & -1 & 0 & 0 & -1/2 \\
x_2 = & 9/2 & 0 & 1 & 0 & -1/2 & 0 & 0 & 0 & 1/2 \\
x_6 = & 13/2 & 0 & 0 & 0 & -1/2 & 1 & 1 & 0 & 1/2 \\
x_7 = & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
x_1 = & 7/2 & 1 & 0 & 0 & -1/2 & 0 & 0 & 0 & -1/2 \\
\end{array}
\]

So the optimal solution is $(7/2, 9/2, 11/2, 0, 0, 13/2, 2)$, with optimal value 8.