Math 373: Mathematical Programming and Optimization I  
Fall, 2018    Assignment 4  
November 11, due November 23

1. A farmer has 25 acres of land on which to grow barley or corn. He earns $600 for each acre of barley and $500 for each acre of corn. Harvesting the barley requires 20 hours of labour per acre, and harvesting the corn requires 10 hours of labour per acre. The farmer can afford 200 hours of labor and wants to maximize his profit.

(a) Write down this linear programming problem in standard form.

Let $x_1$ denote the number of acres of barley to grow and $x_2$ denote the number of acres of corn to grow.

\[
\begin{align*}
\text{minimize} & \quad -600x_1 - 500x_2 \\
\text{subject to} & \quad x_1 + x_2 + x_3 = 25, \\
& \quad 20x_1 + 10x_2 + x_4 = 200, \\
& \quad x_1, x_2, x_3, x_4 \geq 0.
\end{align*}
\]

(b) Determine an initial feasible solution by setting all nonslack variables to zero.

\[
x = (0, 0, 25, 200).
\]

(c) Starting from your solution in (b), carry out Phase II of the simplex method to determine the optimal solution and optimal profit.

\[
\begin{array}{c|cccc}
& x_1 & x_2 & x_3 & x_4 \\
x_3 & 0 & -600 & -500 & 0 & 0 \\
x_4 & 25 & 1 & 1 & 1 & 0 \\
 & 200 & 20 & 10 & 0 & 1 \\
\hline
x_1 & 6000 & 0 & -200 & 0 & 30 \\
x_2 & 15 & 0 & 1/2 & 1 & -1/20 \\
x_3 & 10 & 1 & 1/2 & 0 & 1/20 \\
\hline
x_4 & 10000 & 400 & 0 & 0 & 50 \\
x_1 & 5 & -1 & 0 & 1 & -1/10 \\
x_2 & 20 & 2 & 1 & 0 & 1/10 \\
\end{array}
\]

We thus see that the optimal feasible solution is \((0, 20, 5, 0)\) which corresponds to the solution \((0, 20)\), for an optimal profit of $10,000. That is, the farmer should only plant 20 acres of corn.
2. Suppose you are asked to produce two kinds of meals: economy and deluxe. The economy meal sells for $3/kg and the deluxe version sells for $4/kg. The ingredients are rice and lamb. The economy version should contain at most 25% lamb, while the deluxe version should contain at least 50% lamb. The cost of rice is $1/kg, whereas the lamb costs $2/kg. You have available 300 kg of rice and 100 kg of lamb. Formulate a linear programming problem that investigates how much rice and lamb should you put into each dish to maximize your profit, assuming that all of the prepared dishes sell.

(a) Write down this linear programming problem in standard form.

Let the decision variables $x_1$ and $x_2$, represent the weight of rice in the economy and deluxe dishes respectively, and $x_3$ and $x_4$ represent the corresponding weights of lamb. One wants to maximize the objective function

$$f(x_1, x_2, x_3, x_4) = 3(x_1 + x_3) + 4(x_2 + x_4) - (x_1 + x_2) - 2(x_3 + x_4) = 2x_1 + 3x_2 + x_3 + 2x_4$$

subject to the constraints

$$x_1 + x_2 \leq 300,$$

$$x_3 + x_4 \leq 100,$$

$$x_3 \leq \frac{1}{4}(x_1 + x_3),$$

$$x_4 \geq \frac{1}{2}(x_2 + x_4),$$

and

$$x_1, x_3, x_2, x_4 \geq 0.$$

In standard form, this linear program problem appears as

$$\text{minimize } -2x_1 - 3x_2 - x_3 - 2x_4$$

subject to

$$x_1 + x_2 + x_5 = 300,$$

$$x_3 + x_4 + x_6 = 100,$$

$$-x_1 + 3x_3 + x_7 + x_8 = 0,$$

$$x_2 - x_4 + x_8 = 0,$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0.$$  

(b) Determine an initial feasible solution by setting all nonslack variables to zero.

$$x = (0, 0, 0, 0, 300, 100, 0, 0).$$
(c) Starting from your solution in (b), carry out Phase II of the simplex method to determine the optimal solution and optimal profit.

<table>
<thead>
<tr>
<th>$x_1^*$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_8$</th>
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<tbody>
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<tr>
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<thead>
<tr>
<th>$x_1$</th>
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<tr>
<th>$x_1$</th>
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<td>2</td>
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</table>

So to obtain the maximum profit of $900, you should use put 200 kg of rice and no lamb in the economy dish, and 100 kg of rice and 100 kg of lamb in the deluxe dish.
3. Use the two-phase simplex method to determine the optimal solution and optimal cost of the linear programming problem

\[
\begin{align*}
\text{minimize} & \quad x_1 + 4x_2 \\
\text{subject to} & \quad x_1 - 2x_2 \geq 5, \\
& \quad x_1 - x_2 = 6, \\
& \quad x_1 - 3x_2 \geq -7, \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]

**Phase I:**

**Step 1.** Multiply every constraint with a negative right-hand side by \(-1\), so that \(b \geq 0\):

\[
\begin{align*}
\text{minimize} & \quad x_1 + 4x_2 \\
\text{subject to} & \quad x_1 - 2x_2 \geq 5, \\
& \quad x_1 - x_2 = 6, \\
& \quad -x_1 + 3x_2 \leq 7, \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]

**Step 2.** We introduce the slack variables \(x_3\) and \(x_4\), along with the artificial variables \(x_5, x_6,\) and \(x_7\), to form the auxiliary problem

\[
\begin{align*}
\text{minimize} & \quad x_5 + x_6 + x_7 \\
\text{subject to} & \quad x_1 - 2x_2 - x_3 + x_5 = 5, \\
& \quad x_1 - x_2 + x_6 = 6, \\
& \quad -x_1 + 3x_2 + x_4 + x_7 = 7, \\
& \quad x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0.
\end{align*}
\]

Note: since \(A_4\) is a unit vector, we could have optimized our calculations to avoid introducing the artificial variable \(x_7\).

<table>
<thead>
<tr>
<th>(\hat{x}_5)</th>
<th>(x_1^*)</th>
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<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>(x_6)</th>
<th>(x_7)</th>
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<td>0</td>
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<td>-1</td>
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<tr>
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<td>(x_6)</td>
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<td>-2</td>
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<tr>
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<td>1</td>
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</tbody>
</table>
Phase II:

Thus the optimal solution is (6, 0, 1, 13), which corresponds to the optimal solution (6, 0) to the original problem, with optimal cost 6.

4. Consider the linear programming problem

\[
\begin{align*}
\text{minimize} & \quad x_1 + 2x_2 \\
\text{subject to} & \quad x_1 + 3x_2 \leq 2, \\
& \quad 2x_1 + x_2 = 1, \\
& \quad x_1 \geq 0.
\end{align*}
\]

(a) Put this problem into standard form, labelling the new decision variables with sequential integer indices.
minimize \quad x_1 + 2x_2 - 2x_3 \\
subject to \quad x_1 + 3x_2 - 3x_3 + x_4 = 2, \\
\quad 2x_1 + x_2 - x_3 = 1, \\
\quad x_1, x_2, x_3, x_4 \geq 0.

(b) Use the two-phase simplex method to show that the optimal cost of this problem is \(-\infty\).

Phase I:

**Step 1.** Multiply every constraint with a negative right-hand side by \(-1\), so that \(b \geq 0\). In this case, there is no need to change the constraints.

**Step 2.** In order to find a feasible solution, the artificial variables \(x_5\) and \(x_6\) are introduced to form the auxiliary problem

minimize \quad x_5 + x_6 \\
subject to \quad x_1 + 3x_2 - 3x_3 + x_4 + x_5 = 2, \\
\quad 2x_1 + x_2 - x_3 + x_6 = 1, \\
\quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(x_1^*)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>(x_6)</th>
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<tbody>
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<td>(-4)</td>
<td>(4)</td>
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<tr>
<td>(x_6^\dagger)</td>
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<tr>
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<th>(x_4)</th>
<th>(x_5)</th>
<th>(x_6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_5^\dagger)</td>
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<td>(-5/2)</td>
<td>(5/2)</td>
<td>(-1)</td>
<td>(0)</td>
</tr>
<tr>
<td>(x_1^\dagger)</td>
<td>(3/2)</td>
<td>(0)</td>
<td>(5/2)</td>
<td>(-5/2)</td>
<td>(1)</td>
<td>(1)</td>
</tr>
<tr>
<td>(x_1)</td>
<td>(1/2)</td>
<td>(1)</td>
<td>(1/2)</td>
<td>(-1/2)</td>
<td>(0)</td>
<td>(0)</td>
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<table>
<thead>
<tr>
<th>(x)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>(x_6)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(1)</td>
<td>(1)</td>
</tr>
<tr>
<td>(x_2)</td>
<td>(3/5)</td>
<td>(0)</td>
<td>(1)</td>
<td>(-1)</td>
<td>(2/5)</td>
<td>(2/5)</td>
</tr>
<tr>
<td>(x_1)</td>
<td>(1/5)</td>
<td>(1)</td>
<td>(0)</td>
<td>(0)</td>
<td>(-1/5)</td>
<td>(-1/5)</td>
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Phase II:

<table>
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<tr>
<th>(x)</th>
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<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_2^\dagger)</td>
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<td>(0)</td>
<td>(0)</td>
</tr>
<tr>
<td>(x_2^\dagger)</td>
<td>(3/5)</td>
<td>(0)</td>
<td>(1)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(x_1)</td>
<td>(1/5)</td>
<td>(1)</td>
<td>(0)</td>
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</table>
Since all entries in column $x_3$ are negative, the simplex method terminates with optimal cost $-\infty$.

(c) For each $t \geq 0$, use the final simplex iteration to determine a feasible solution $x(t)$ such that $\lim_{t \to \infty} c^t x(t) = -\infty$.

Let $x_0 = (1/2, 0, 0, 3/2)$ be the final vertex encountered in Phase II. For the 3rd basic direction, we deduce from $d_B = -B^{-1}A_3 = (5/2, 1/2)$ that $d = (1/2, 0, 1, 5/2)$.

$$x(t) = x_0 + td = \left(\frac{1}{2} + \frac{1}{2} t, 0, t, \frac{3}{2} + \frac{5}{2} t\right).$$

Using the cost vector $c(t) = (1, 2, -2, 0)$ from the standard-form problem, we see that the optimal cost is

$$\lim_{t \to \infty} c^t x(t) = \lim_{t \to \infty} \frac{1}{2} + \frac{1}{2} t - 2t = \lim_{t \to \infty} \frac{1}{2} - \frac{3}{2} t = -\infty.$$