1. Use Theorem 2.7 to establish that the system of inequalities
\[ \begin{align*}
  x_1 - 2x_2 & \geq 2, \\
  2x_1 + 3x_2 & \leq 1
\end{align*} \]
has no non-negative solutions.

2. This standard-form linear programming problem has 3 basic feasible solutions:
\[
\begin{align*}
\text{minimize} & \quad x_1 + x_2 + x_3 \\
\text{subject to} & \quad 2x_1 - x_2 + x_3 = 6, \\
& \quad -x_1 + x_2 + x_4 = -2, \\
& \quad x_1, x_2, x_3, x_4 \geq 0.
\end{align*}
\]
(a) Compute \( e_B^T B^{-1} \) for the basis indices below corresponding to the 3 basic feasible solutions. There is no need here to actually compute the basic solutions.
\[
\begin{align*}
B(1) = 1 \text{ and } B(2) = 2 \\
B(1) = 1 \text{ and } B(2) = 3 \\
B(1) = 1 \text{ and } B(2) = 4
\end{align*}
\]
(b) Use your results in part (a) to determine the reduced cost \( c_j = c_j - e_B^T B^{-1} A_j \) for each nonbasic variable \( x_j \) of each basic feasible solution.
(c) Determine the optimal feasible solution and optimal value from part (b). Check your work by verifying that your optimal solution satisfies the constraints.

3. (a) Find all vertices of the polyhedron
\[ P = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 = 4, x_3 = 2, x_2 \geq 1, x_1, x_2, x_3 \geq 0 \right\}. \]
Show your work. Do not solve this problem graphically.
(b) Determine which of the vertices are degenerate, and explain how you reached this conclusion.
(c) Consider the linear programming problem of minimizing \( c^T x \) where \( c = (1, -2, 3) \), over the above polyhedron \( P \). Is there an optimal solution? If yes, explain why and determine the possible optimal solutions and corresponding optimal cost. If no, explain why not. Hint: is \( P \) bounded?
4. Consider the linear programming problem

\[
\begin{align*}
\text{minimize} & \quad x_1 + 2x_2 + x_3 \\
\text{subject to} & \quad 3x_1 + 3x_2 + x_3 \geq 3, \\
& \quad x_1 + x_2 + x_3 \leq 2, \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}
\]

Let \( S \) be the feasible set.

(a) Sketch \( S \).

(b) Locate the extreme points of \( S \) graphically.

(c) Compute the value of the objective function at each of the extreme points of \( S \).

(d) Find an optimal solution to this linear programming problem.

(e) Put this linear programming problem into standard form and find a basis matrix \( B \) that corresponds to the optimal solution you found in part (d).

(f) Compute the reduced cost vector \( \bar{c} \) at the optimal solution you found in part (d).