1. Let \( x \) be an element of the polyhedron \( P = \{ x \in \mathbb{R}^n : Ax = b, x \geq 0 \} \). Prove that a vector \( d \in \mathbb{R}^n \) is a feasible direction at \( x \) if and only if \( Ad = 0 \) and \( d_i \geq 0 \) for every \( i \) such that \( x_i = 0 \).

If \( x \) is a feasible direction, then \( x + td \in P \) for some positive scalar \( t \). That is, \( A(x + td) = b \) and \( x + td \geq 0 \). Then \( tAd = A(x + td) - Ax = b - b = 0 \), so that \( Ad = 0 \). Moreover, for each zero component \( x_i \), the condition \( x_i + td_i \geq 0 \) reduces to \( d_i \geq 0 \).

Conversely, if there exists a direction such that \( Ad = 0 \) and \( d_i \geq 0 \) for every \( i \) such that \( x_i = 0 \), then \( A(x + td) = Ax + tAd = b + 0 = b \) for every real \( t \). On choosing \( t^* = \min_{x_i > 0, d_i \neq 0} x_i/|d_i| > 0 \), we thus see that \( x + t^*d \geq 0 \). Hence \( d \) is a feasible direction at \( x \).

2. Let \( x \) be a basic feasible solution of a linear programming problem \( \Pi \) written in standard form, with associated basis matrix \( B \) and set of nonbasic indices \( N \). Let \( y \) be any feasible solution to \( \Pi \) and consider the difference vector \( d = y - x \).

(a) Prove that \( d_j \geq 0 \) for every \( j \in N \).

For any feasible solution \( y \) we have \( y \geq 0 \). Since \( x \) is a basic feasible solution, we know for each \( j \in N \) that \( x_j = 0 \) and hence \( d_j = y_j - x_j \geq 0 \).

(b) If \( d_j = 0 \) for every \( j \in N \), prove that \( y = x \).

This would imply that
\[
0 = Ay - Ax = Ad = Bd_B + \sum_{j \in N} A_j d_j = Bd_B.
\]

The linear independence of the columns of \( B \) then implies that \( d_B = 0 \) and hence \( d = 0 \), so that \( y = x \).

(c) If the reduced cost \( c_j \) of every nonbasic variable \( x_j \) is positive, use parts (a) and (b) to prove that \( x \) is the unique optimal solution to \( \Pi \).

Recall that \( c_j \) is the rate of change along the \( j \)th simplex direction. That is, the change in cost on moving from \( x \) to \( y \) is
\[
c^\top y - c^\top x = c^\top d = c^\top_B d_B + \sum_{j \in N} c_j d_j = \sum_{j \in N} (c_j - c^\top_B B^{-1} A_j) d_j = \sum_{j \in N} c_j d_j.
\]
We know from part (a) that \( d_j \geq 0 \). Moreover, if \( y \neq x \), we know from part (b) that \( d_j > 0 \) for some \( j \in N \). Given \( \tilde{c}_j > 0 \) for each \( j \in N \), we see that

\[
c^\top y - c^\top x = \sum_{j \in N} \tilde{c}_j d_j > 0.
\]

Since this holds for every feasible vector \( y \neq x \), we see that \( x \) is the unique optimal solution.

(d) Suppose that \( \Pi \) is nondegenerate and that \( x \) is an optimal solution to \( \Pi \). If the reduced cost \( \tilde{c}_j \) of some nonbasic variable \( x_j \) is zero, prove that \( \Pi \) does not have a unique optimal solution.

Let \( d' \) be the \( j \)th simplex direction. Since the problem is nondegenerate, we know that the solution \( y = x + td' \) is feasible for some \( t > 0 \). From the definition of the \( j \)th simplex direction, we see that

\[
c^\top y - c^\top x = t\tilde{c}_j d'_j = 0.
\]

That is, \( y \) is a distinct feasible solution with the same optimal cost as \( x \).