1. Use complementary slackness to determine the set \( \{c_1, c_2, c_3\} \) for which \((1, 1, 0)\) is an optimal solution to the linear programming problem

\[
\begin{align*}
\text{minimize} & \quad c_1 x_1 + c_2 x_2 + c_3 x_3 \\
\text{subject to} & \quad x_1 - x_2 + x_3 \leq 0, \\
& \quad 2x_1 + x_2 - x_3 = 3, \\
& \quad x_1 - x_2 - x_3 \leq 1, \\
& \quad [x_1, x_2, x_3] \geq 0.
\end{align*}
\]

The dual problem is

\[
\begin{align*}
\text{maximize} & \quad 3p_2 + p_3 \\
\text{subject to} & \quad p_1 + 2p_2 + p_3 \leq c_1, \\
& \quad -p_1 + p_2 - p_3 \leq c_2, \\
& \quad p_1 - p_2 - p_3 \leq c_3, \\
& \quad p_1 \leq 0, \quad p_3 \leq 0.
\end{align*}
\]

Since only the first two constraints of the primal problem are active at \( x = (1, 1, 0) \), we require \( p_3 = 0 \). Moreover, since \( x_1 \) and \( x_2 \) are nonzero, the first two dual constraints must also be active:

\[
\begin{align*}
p_1 + 2p_2 &= c_1, \\
-p_1 + p_2 &= c_2.
\end{align*}
\]

On adding these equations, we find that \( p_2 = (c_1 + c_2)/3 \). We require that

\[
0 \geq p_1 = p_2 - c_2 = c_1/3 - 2c_2/3,
\]

which implies that \( c_1 \leq 2c_2 \). We also require that \( x \) and \( p \) be feasible solutions. The given solution \((1, 1, 0)\) satisfies all six primal constraints (including the sign conditions). For the dual solution \((p_1, p_2, 0)\) to satisfy all six dual constraints, we require in addition that \( p_1 - p_2 \leq c_3 \), which reduces to \(-c_2 \leq c_3 \). The complementary slackness theorem then guarantees that the solution set of cost coefficients such that \((1, 1, 0)\) is an optimal solution to the primal problem is

\[
\{(c_1, c_2, c_3) : c_1 \leq 2c_2, c_3 \geq -c_2\}.
\]
2. If a linear programming problem in standard form has a non-degenerate basic feasible solution that is optimal, prove that the dual problem has a unique optimal solution. Hint: consider complementary slackness.

Let \( x^* \) be a non-degenerate basic optimal solution to the primal problem. Since the primal problem has an optimal solution, the dual has an optimal solution \( p \). Let \( j_1, \ldots, j_m \) be a set of basic indices corresponding to \( x^* \) and consider the complementary slackness condition

\[
(c_j - p^T A_j) x^*_j = 0, \quad j = j_1, \ldots, j_m.
\]

Since \( x^* \) is nondegenerate, we know that \( x^*_j > 0 \) for each basic variable \( x_j \), so that

\[
c_j = p^T A_j, \quad j = j_1, \ldots, j_m.
\]

This is just the system of equations

\[
c^T_B = p^T B,
\]

which has a unique solution \( p^T = c^T_B B^{-1} \) since the basis \( B = \{ A_j : j = j_1, \ldots, j_m \} \) is invertible.