Math 373: Mathematical Programming and Optimization I
Fall, 2023 Assignment 1
September 18, due October 2

1. Suppose that \((x_1, x_2, x_3)\) is a feasible solution to the linear programming problem

\[
\begin{align*}
\text{minimize} & \quad 4x_1 + 2x_2 + x_3 \\
\text{subject to} & \quad x_1 - x_2 \geq 3, \\
& \quad 2x_1 + x_2 + x_3 \geq 4, \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}
\]

Let \(y_1\) and \(y_2\) be non-negative numbers.

(a) Show that

\[
x_1(y_1 + 2y_2) + x_2(-y_1 + y_2) + x_3y_2 \geq 3y_1 + 4y_2.
\]

(b) Find constraints on \(y_1\) and \(y_2\) so that

\[
4x_1 + 2x_2 + x_3 \geq x_1(y_1 + 2y_2) + x_2(-y_1 + y_2) + x_3y_2
\]

at every feasible solution \((x_1, x_2, x_3)\).

(c) Use parts (a) and (b) to find a lower bound to the optimal cost in terms of only the variables \(y_1\) and \(y_2\).

(d) Formulate the linear programming problem in the variables \((y_1, y_2)\) that determines the largest possible value for the lower bound to the optimal cost found in part (c).

2. Let \(g : \mathbb{R} \to \mathbb{R}\) be a convex function on an interval \(I\). Let \(f : \mathbb{R} \to \mathbb{R}\) be a convex increasing (but not necessarily differentiable) function on \(\mathbb{R}\). Show that \(f \circ g\) is convex on \(I\).

3. Let \(A\) be an \(m \times n\) matrix and let \(b \in \mathbb{R}^m\). Prove that exactly one of these alternatives holds:

(i) \(Ax = b\) has a solution \(x\);
(ii) \(A^tp = 0\) has a solution \(p\) with \(p^tb \neq 0\).