1. Consider the linear programming problem

\[
\begin{align*}
\text{maximize} & \quad x_1 + x_2 \\
\text{subject to} & \quad ax_1 + bx_2 \leq c, \\
& \quad x_1, x_2 \geq 0,
\end{align*}
\]

where \(a, b, c \in \mathbb{R}\).

(a) Provide a triple \((a, b, c)\) such that the problem does not have a feasible solution.

For example: \((a, b, c) = (1, 1, -1)\).

(b) Provide a triple \((a, b, c)\) such that the problem has a unique optimal solution.

For example: \((a, b, c) = (1, 1, 0)\).

(c) Provide a triple \((a, b, c)\) such that the optimal solution is unbounded.

For example: \((a, b, c) = (-1, -1, 0)\).

(d) Provide a triple \((a, b, c)\) such that the problem has infinitely many optimal solutions.

For example: \((a, b, c) = (1, 1, 1)\).

2. Consider the linear programming problem:

\[
\begin{align*}
\text{minimize} & \quad x_1 - 2x_2 + 3x_3 \\
\text{subject to} & \quad x_1 + x_2 + x_3 \leq 1, \\
& \quad x_1 + 2x_2 + 4x_3 = 3, \\
& \quad x_2 \geq 0, \\
& \quad x_3 \geq 0.
\end{align*}
\]

(a) Write down an equivalent version of the above linear programming problem.

\[
\begin{align*}
\text{minimize} & \quad x_1^+ - x_1^- - 2x_2 + 3x_3 \\
\text{subject to} & \quad x_1^+ - x_1^- + x_2 + x_3 + x_4 = 1, \\
& \quad x_1^+ - x_1^- + 2x_2 + 4x_3 = 3, \\
& \quad x_1^+ \geq 0, \\
& \quad x_1^- \geq 0, \\
& \quad x_2 \geq 0, \\
& \quad x_3 \geq 0, \\
& \quad x_4 \geq 0.
\end{align*}
\]
(b) Is the solution \((0, \frac{1}{2}, \frac{1}{2})\) of the original problem feasible? Yes.

(c) What is the cost for this solution? 0.5

(d) Find a corresponding solution to your standard-form problem.

\[
(x_1^+, x_1^-, x_2, x_3, x_4) = \left(0, 0, \frac{1}{2}, \frac{1}{2}, 0\right).
\]

(e) Is this solution to your standard-form problem feasible? Yes.

(f) What is the cost for your standard-form problem? 0.5

3. Consider a two-player game \(G\) with the following payoff matrix pair

\[
A = \begin{pmatrix} 3 & 0 \\ 2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 \\ 5 & 5 \end{pmatrix}.
\]

(a) Denote the strategy of the first player by \(y\) and of the second player by \(z\). Analytically determine the set of Nash equilibria of the game. Write down the corresponding linear programming problems.

We have

\[
y = \begin{bmatrix} y_1 \\ 1 - y_1 \end{bmatrix}, \quad y_1 \geq 0, \quad z = \begin{bmatrix} z_1 \\ 1 - z_1 \end{bmatrix}, \quad z_1 \geq 0.
\]

A Nash equilibrium \((y, z)\) must satisfy

\[
\forall x, \quad u_1(y, z) \geq u_1(x, z), \quad (2)
\]

\[
\forall x, \quad u_2(y, z) \geq u_2(y, x), \quad (3)
\]

Condition (2) implies that \(y\) must maximize \(u_1(x, z)\) over \(x\), and condition (3) implies that \(z\) must maximize \(u_2(y, x)\) over \(x\). We find

\[
y = \arg \max_{x \in \Delta_2} x^T A z = \arg \max_{x_1} \left[ x_1 \begin{bmatrix} 3 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ 1 - z_1 \end{bmatrix} \right] = \arg \max_{x_1} \left[ x_1 \begin{bmatrix} 1 - x_1 \\ z_1 \end{bmatrix} \right] = \begin{bmatrix} 3z_1 \\ 2 \end{bmatrix}.
\]
So based on the ordering of \(3z_1\) and \(2\), \(y_1\) is given by
\[
\begin{align*}
 &\begin{cases}
y_1 = 1 & z_1 > \frac{2}{3} \\
y_1 \in [0, 1] & z_1 = \frac{2}{3} \\
y_1 = 0 & z_1 < \frac{2}{3}
\end{cases} \\
\end{align*}
\tag{4}
\]
Likewise,
\[
z = \arg\max_{y_1} y^T B x = \arg\max_{x_1} \left[ y_1 \begin{bmatrix} 1 - y_1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ 1 - x_1 \end{bmatrix} \right] = \arg\max_{x_1} \left[ 5 - 2y_1 \right] \begin{bmatrix} 5 - 5y_1 \\ 1 - x_1 \end{bmatrix}.
\]
So based on the ordering of \(5 - 2y_1\) and \(5 - 5y_1\), \(z_1\) is given by
\[
\begin{align*}
 &\begin{cases}
z_1 = 1 & 5 - 2y_1 > 5 - 5y_1 \\
z_1 \in [0, 1] & 5 - 2y_1 = 5 - 5y_1 \\
z_1 = 0 & 5 - 2y_1 < 5 - 5y_1
\end{cases} \\
\end{align*}
\tag{5}
\]
Now (4) and (5) result in the following possibilities
\[
\begin{align*}
 &\begin{cases}
y_1 = 1, z_1 = 1 \\
z_1 = \frac{2}{3}, y_1 = 0 \\
y_1 = 0, z_1 < \frac{2}{3}
\end{cases} \Rightarrow \\
 &\begin{cases}
y_1 = 1, z_1 = 1 \\
y_1 = 0, z_1 \leq \frac{2}{3}
\end{cases} \Rightarrow \\
 &\begin{cases}
y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, z = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, z = \begin{bmatrix} 1 \\ 1 - z_1 \end{bmatrix}
\end{cases}, \quad 0 \leq z_1 \leq \frac{2}{3}.
\end{align*}
\]

The corresponding linear programming problems are, given \((z_1, z_2)\):
\[
\begin{align*}
\text{maximize} & \quad 3z_1x_1 + 2x_2 \\
\text{subject to} & \quad x_1 + x_2 = 1, \\
& \quad x_1, x_2 \geq 0,
\end{align*}
\]
and, given \((y_1, y_2)\):
\[
\begin{align*}
\text{maximize} & \quad (5 - 2y_1)x_1 + 5(1 - y_1)x_2 \\
\text{subject to} & \quad x_1 + x_2 = 1, \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]

(b) Consider another two-player game \(G'\) with payoff matrices
\[
A' = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \quad B' = \begin{pmatrix} 6 & 0 \\ 10 & 10 \end{pmatrix}.
\tag{6}
\]
Using part (a), is it possible to make a statement about the set of Nash equilibria of \(G'\) by just comparing the payoff matrices in Eqs. (1) and (6)? Explain.
Adding a constant to all entries of any column of $A$ does not change the set of Nash equilibria:

$$\arg \max_{x \in \Delta^2} \left( \sum_{i,j} x_i (A_{ij} + c_j) z_j \right) = \arg \max_{x \in \Delta^2} \left( \sum_{i,j} x_i A_{ij} z_j + \sum_i x_i \sum_j c_j z_j \right)$$

$$= \arg \max_{x \in \Delta^2} \left( \sum_{i,j} x_i A_{ij} z_j + \sum_j c_j z_j \right)$$

$$= \arg \max_{x \in \Delta^2} \sum_{i,j} x_i A_{ij} z_j.$$ 

The same holds true when adding a constant to all entries of any row of $B$ or when a payoff matrix is multiplied by a constant.

Now if we add $-3$ to the first column of $A$ and $-1$ to the second column of $A$, we get $A'$. Moreover, if we multiply $B$ by 2, we get $B'$. Hence, $G$ and $G'$ have the same Nash equilibria.