Theorem 5.1.6. *Let $\varnothing \neq U \subset \mathbb{R}^N$ be open, $N \geq M$, and $f \in \mathcal{C}^1(U, \mathbb{R}^M)$ with* rank $J_f(x) = M$ *for $x \in U$. Then $f(U)$ is open.*

Theorem 5.2.5 (Inverse Function Theorem)**.** *Let $\varnothing \neq U \subset \mathbb{R}^N$ be open, let $f \in \mathcal{C}^1(U, \mathbb{R}^N)$, and let $x_0 \in U$ be such that $\det J_f(x_0) \neq 0$. Then there exists an open neighborhood $V \subset U$ of x_0 such that f is injective on V, $f(V)$ is open, and $f^{-1} \colon f(V) \to \mathbb{R}^N$ is a $\mathcal{C}^1$-function such that $J_{f^{-1}} = J_f^{-1}$.*

Theorem 5.2.6 (Implicit Function Theorem)**.** *Let $\varnothing \neq U \subset \mathbb{R}^{M+N}$ be open, let $f \in \mathcal{C}^1(U, \mathbb{R}^N)$, and let $(x_0, y_0) \in U$ be such that $f(x_0, y_0) = 0$ and $\det \frac{\partial f}{\partial y}(x_0, y_0) \neq 0$. Then there exist neighborhoods $V \subset \mathbb{R}^M$ of x_0 and $W \subset \mathbb{R}^N$ of y_0 with $V \times W \subset U$ and a unique $\phi \in \mathcal{C}^1(V, \mathbb{R}^N)$ such that:*

(i) *$\phi(x_0) = y_0$;*

(ii) *$f(x, y) = 0$ if and only if $\phi(x) = y$ for all $(x, y) \in V \times W$.*

Moreover, we have

$$J_\phi = -\left(\frac{\partial f}{\partial y}\right)^{-1} \frac{\partial f}{\partial x}.$$

Theorem 5.3.2 (Lagrange Multiplier Theorem)**.** *Let $N \geq 2$, let $\varnothing \neq U \subset \mathbb{R}^N$ be open, let $f, \phi \in \mathcal{C}^1(U, \mathbb{R})$, and let $x_0 \in U$ be such that f has a local extremum at x_0 under the constraint $\phi(x) = 0$ and such that $\nabla \phi(x_0) \neq 0$. Then there exists $\lambda \in \mathbb{R}$, a* Lagrange multiplier*, such that*

$$\nabla f(x_0) = \lambda \nabla \phi(x_0).$$

Theorem 5.4.1 (Change of Variables)**.** *Let $\varnothing \neq U \subset \mathbb{R}^N$ be open, let $\varnothing \neq K \subset U$ be compact with content, let $\phi \in \mathcal{C}^1(U, \mathbb{R}^N)$, and suppose that there exists a set $Z \subset K$ with content zero such that $\phi|_{K \setminus Z}$ is injective and $\det J_\phi(x) \neq 0$ for all $x \in K \setminus Z$. Then $\phi(K)$ has content and*

$$\int_{\phi(K)} f = \int_K (f \circ \phi) |\det J_\phi|$$

holds for all continuous functions $f \colon \phi(U) \to \mathbb{R}^M$.

Theorem 6.2.7 (Fundamental Theorem for Curve Integrals)**.** *Let $\varnothing \neq U \subset \mathbb{R}^N$ be open, let $f \colon U \to \mathbb{R}^N$ be a continuous, conservative vector field with potential function $F \colon U \to \mathbb{R}$ and let $\gamma \colon [a, b] \to U$ be a piecewise $\mathcal{C}^1$ curve. Then*

$$\int_\gamma f \cdot dx = F(\gamma(b)) - F(\gamma(a)).$$

Theorem 6.2.10. *Let $\varnothing \neq U \subset \mathbb{R}^N$ be open and convex, and let $f : U \to \mathbb{R}^N$ be continuous. Then the following are equivalent:*

(i) *f is conservative;*

(ii) *$\int_\gamma f \cdot dx = 0$ for each closed, piecewise $\mathcal{C}^1$ curve γ in U.*

Theorem 6.2.14. *Let $f\colon [a,b] \times [c,d] \to \mathbb{R}$ be continuous, and suppose further that $\dfrac{\partial f}{\partial x}$ exists and is continuous on $[a,b] \times [c,d]$. Define*

$$F\colon [a,b] \to \mathbb{R}, \quad x \mapsto \int_c^d f(x,y)\,dy.$$

Then F is continuously differentiable, with

$$F'(x) = \int_c^d \frac{\partial f}{\partial x}(x,y)\,dy$$

for $x \in [a,b]$.

Theorem 6.2.15. *Let $\varnothing \neq U \subset \mathbb{R}^N$ be open, and let $f : U \to \mathbb{R}^N$ be a $\mathcal{C}^1$ vector field. Consider the following statements:*

(i) *f is conservative;*

(ii) *f satisfies*

$$\frac{\partial f_j}{\partial x_k} = \frac{\partial f_k}{\partial x_j}$$

Then (i) $\Longrightarrow$ (ii), *and* (ii) $\Longrightarrow$ (i) *if there exists $x_0 \in U$ such that $[x_0, x] \subset U$ for all $x \in U$.*

Theorem 6.3.6 (Green's Theorem)**.** *Let $\varnothing \neq U \subset \mathbb{R}^2$ be open, let $K \subset U$ be a normal domain, and let $P, Q \in \mathcal{C}^1(U, \mathbb{R})$. Then*

$$\int_K \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \int_{\partial K} P\,dx + Q\,dy.$$

Theorem 6.6.5 (Stokes' Theorem)**.** *Suppose that the following hypotheses are given:*

(a) *Φ is a $\mathcal{C}^2$ surface for which the parameter domain K is a normal domain (with respect to both axes);*

(b) *the positively oriented boundary ∂K of K is parametrized by a piecewise $\mathcal{C}^1$ curve $\gamma\colon [a,b] \to \mathbb{R}^2$;*

(c) *P, Q, and R are $\mathcal{C}^1$-functions defined on an open set containing $\{\Phi\}$.*

Then

$$\begin{aligned}
&\int_{\Phi\circ\gamma} P\,dx + Q\,dy + R\,dz \\
&= \int_{\Phi} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy \\
&= \int_{\Phi} (\text{curl } f) \cdot n\,d\sigma,
\end{aligned}$$

where $f = (P, Q, R)$.

Theorem 6.7.4 (Gauß' Theorem)**.** *Let $U \subset \mathbb{R}^3$ be open, let $V \subset U$ be a normal domain with boundary S, and let $f \in \mathcal{C}^1(U, \mathbb{R}^3)$. Then*

$$\int_S f \cdot n\,d\sigma = \int_V \text{div } f.$$

Theorem 7.5.2 (Stokes' Theorem for Differential Forms over r-Chains)**.** *Let Φ be an r-chain, and let ω be an $(r-1)$-form of class $\mathcal{C}^1$ on an open neighborhood of $\{\Phi\}$. Then*

$$\int_{\Phi} d\omega = \int_{\partial\Phi} \omega.$$

Theorem 8.1.22 (Riemann's Rearrangement Theorem)**.** *Let $\sum_{k=1}^{\infty} a_k$ be convergent, but not absolutely convergent, and let $x \in \mathbb{R}$. Then there exists a bijective map $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sum_{k=1}^{\infty} a_{\sigma(k)} = x$.*

Theorem 8.1.23 (Cauchy Product Formula)**.** *Suppose that $\displaystyle\sum_{k=0}^{\infty} a_k$ and $\displaystyle\sum_{k=0}^{\infty} b_k$ converge absolutely. Then $\displaystyle\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k}$ converges absolutely such that*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_n b_{n-k} = \left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right).$$

Theorem 9.1.3. *Let $\varnothing \neq D \subset \mathbb{R}^N$, and let $f, f_1, f_2, \ldots$ be functions on D such that $f_n \to f$ uniformly on D and such that $f_1, f_2, \ldots$ are continuous. Then f is continuous.*

Corollary 9.1.4. *Let $\varnothing \neq D \subset \mathbb{R}^N$ have content, and let $(f_n)_{n=1}^\infty$ be a sequence of continuous functions on D that converges uniformly on D to $f\colon D \to \mathbb{R}$. Then*

$$\int_D f = \lim_{n\to\infty} \int_D f_n.$$

Theorem 9.1.5. *Let $a < b$, and let $(f_n)_{n=1}^\infty$ be a sequence of continuously differentiable functions on $[a, b]$ such that:*

(a) *$(f_n(x_0))_{n=1}^\infty$ converges at some $x_0 \in [a, b]$;*

(b) *$(f_n')_{n=1}^\infty$ is uniformly convergent.*

Then there exists a continuously differentiable function $f : [a, b] \to \mathbb{R}$ such that $f_n \to f$ and $f_n' \to f'$ uniformly on $[a, b]$.

Theorem 9.1.7. *Let $\varnothing \neq D \subset \mathbb{R}^N$, and let $(f_n)_{n=1}^\infty$ be a sequence of $\mathbb{R}$-valued functions on D. Then the following are equivalent:*

(i) *there exists a function $f\colon D \to \mathbb{R}$ such that $f_n \to f$ uniformly on D;*

(ii) *$(f_n)_{n=1}^\infty$ is a uniform Cauchy sequence on D.*

Theorem 9.1.8 (Weierstraß M-Test)**.** *Let $\varnothing \neq D \subset \mathbb{R}^N$, let $(f_k)_{k=1}^\infty$ be a sequence of $\mathbb{R}$-valued functions on D, and suppose that, for each $k \in \mathbb{N}$, there exists $M_k \geq 0$ such that $|f_k(x)| \leq M_k$ for $x \in D$ and such that $\sum_{k=1}^\infty M_k < \infty$. Then $\sum_{k=1}^\infty f_k$ converges uniformly and absolutely on D.*

Corollary 9.2.4. *Let $\sum_{n=0}^\infty a_n(x - x_0)^n$ be a power series with radius of convergence $R > 0$. Then $\sum_{n=0}^\infty a_n(x - x_0)^n$ converges, for each $r \in (0, R)$, uniformly and absolutely on $[x_0 - r, x_0 + r]$ to a $\mathcal{C}^1$-function $f\colon (x_0 - R, x_0 + R) \to \mathbb{R}$ with first derivative*

$$f'(x) = \sum_{n=1}^\infty n a_n (x - x_0)^{n-1}$$

for $x \in (x_0 - R, x_0 + R)$. Moreover, $F\colon (x_0 - R, x_0 + R) \to \mathbb{R}$ given by

$$F(x) := \sum_{n=0}^\infty \frac{a_n}{n+1}(x - x_0)^{n+1}$$

for $x \in (x_0 - R, x_0 + R)$ is an antiderivative of f.

Proposition 9.2.5 (Cauchy–Hadamard Formula)**.** *The radius of convergence R of the power series $\sum_{n=0}^\infty a_n(x - x_0)^n$ is given by*

$$R = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}},$$

where the convention applies that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

Corollary 9.2.7. *Let f be a function with a power series expansion $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ about $x_0 \in \mathbb{R}$. Then f is infinitely often differentiable on an open interval about x_0 such that*

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

for all $n \in \mathbb{N}_0$. In particular, the power series expansion of f about x_0 is unique.

Theorem 9.2.9 (Abel's Theorem)**.** *Suppose that the series $\displaystyle\sum_{n=0}^{\infty} a_n$ converges. Then the power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges pointwise on $(-1,1]$ to a continuous function.*

Lemma 9.3.11 (Riemann–Lebesgue Lemma)**.** *For $f \in \mathcal{PC}_{2\pi}(\mathbb{R})$, we have that*

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(t) \sin\left(\left(n+\frac{1}{2}\right)t\right) dt = 0.$$

Theorem 9.3.13. *Let $f \in \mathcal{PC}_{2\pi}(\mathbb{R})$ and suppose that f has left- and right-hand derivatives at $x \in \mathbb{R}$. Then*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)) = \frac{1}{2}(f(x^+) + f(x^-)).$$

Theorem 9.3.15. *Let $f \in \mathcal{PC}_{2\pi}(\mathbb{R})$ be continuous and piecewise continuously differentiable on $[-\pi,\pi]$. Then*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$$

converges uniformly to $f(x)$ on $\mathbb{R}$.

Theorem 9.3.18. *Let $f \in \mathcal{PC}_{2\pi}(\mathbb{R})$. Then $\lim_{n\to\infty} \|f - S_n(f)\|_2 \to 0$.*

Corollary 9.3.19 (Parseval's Identity)**.** *Let $f \in \mathcal{PC}_{2\pi}(\mathbb{R})$ have the Fourier coefficients $a_0, a_1, a_2 \ldots, b_1, b_2, \ldots$. Then*

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\|f\|_2^2.$$