1. Let \( f : [a, b] \to \mathbb{R} \) be a \( C^1 \)-function. Show that the graph of \( f \) can be parametrized as a rectifiable curve with length
\[
\int_a^b \sqrt{1 + f'(t)^2} \, dt.
\]

2. Show that \( K := \{(x, y) \in \mathbb{R}^2 : x, y, x + y \in [0, 1]\} \) is a normal domain (with respect to each coordinate axis).

3. In this problem we compute the hypersurface area \( S_n(r) \) of the \( n \)-sphere
\[
S^n(r) = \{(x_1, x_2, \ldots, x_{n+1}) : x_1^2 + \ldots + x_{n+1}^2 = r^2\}
\]
by parameterizing the half-sphere \( \Phi = S^n \cap (\mathbb{R}^n \times [0, \infty)) \) as
\[
\Phi(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n, \sqrt{r^2 - x_1^2 - x_2^2 - \ldots - x_n^2})
\]
and using the expression
\[
2^n \int_0^r \int_0^{\sqrt{r^2-x_1^2}} \cdots \int_0^{\sqrt{r^2-x_1^2-x_2^2-\ldots-x_{n-1}^2}} |N| \, dx_n \ldots dx_2 dx_1
\]
for the hypersurface area of \( \Phi \), where
\[
|N| = \left( \frac{1}{n!} \sum_{i_1=1}^{n+1} \sum_{i_2=1}^{n+1} \cdots \sum_{i_n=1}^{n+1} \left| \frac{\partial (\Phi_{i_1}, \Phi_{i_2}, \ldots, \Phi_{i_n})}{\partial (x_1, x_2, \ldots, x_n)} \right|^2 \right)^{1/2}
\]
is the magnitude of the normal vector \( N \) to \( \Phi \).

(a) What are \( S_1(1) \) and \( S_2(1) \)?

(b) Show that \( S_n(r) = r^n S_n(1) \) for \( r > 0 \).

(c) Show that \( |N| \) simplifies to
\[
|N| = \left( 1 + \sum_{j=1}^{n} \left| \frac{\partial (\Phi_1, \Phi_2, \ldots, \Phi_{j-1}, \Phi_{j+1}, \ldots, \Phi_{n+1})}{\partial (x_1, x_2, \ldots, x_n)} \right|^2 \right)^{1/2}.
\]
(d) Evaluate $|N(x_1, \ldots x_n)|$.

(e) For $n \geq 3$, use part (d) to show that $S_n(1)$ may be expressed as

$$S_n(1) = 4 \int_0^1 \int_0^{\sqrt{1-x_1^2}} \frac{S_{n-2}(\sqrt{1-x_1^2-x_2^2})}{\sqrt{1-x_1^2-x_2^2}} \, dx_2 \, dx_1.$$ 

Then compute $S_n(1)$ in terms of $S_{n-2}(1)$. Hint: once you have applied part (d) consider changing the variables $(x_1, x_2)$ to a more convenient coordinate system.

(f) Use induction to show that

$$S_n(1) = \begin{cases} \frac{2\pi^m}{(m-1)!} & \text{if } n = 2m - 1, \ m \in \mathbb{N}, \\ \frac{(4\pi)^m(m-1)!}{(2m-1)!} & \text{if } n = 2m, \ m \in \mathbb{N}. \end{cases}$$

(g) Deduce the astonishing result that $\lim_{n \to \infty} S_n(1) = 0.$

4* (a) Let $a < b$, and let $f \in C^1([a,b],[\mathbb{R}])$ such that $f \geq 0$. Viewing the graph of $f$ as a subset of the $xy$ plane in $\mathbb{R}^3$ and rotating it about the $x$ axis generates a surface in $\mathbb{R}^3$, a so called rotation surface. Show that the area of this surface is

$$2\pi \int_a^b f(t) \sqrt{1 + f'(t)^2} \, dt.$$ 

(b) What is the area of the sloped surface of a cone having a circular base with radius $r$ and height $h$?