1. Let \( f : [a, b] \to \mathbb{R} \) be a \( C^1 \)-function. Show that the graph of \( f \) can be parametrized as a rectifiable curve with length
\[
\int_a^b \sqrt{1 + f'(t)^2} \, dt.
\]

The graph of \( f \) is naturally parametrized as
\[
\gamma : [a, b] \to \mathbb{R}^2, \quad t \mapsto (t, f(t)).
\]

Since \( f \) is a \( C^1 \)-function, \( \gamma \) is a \( C^1 \)-curve and therefore rectifiable. Moreover, we have
\[
\gamma'(t) = (1, f'(t))
\]
for \( t \in [a, b] \) and thus
\[
|\gamma'(t)| = \sqrt{1 + f'(t)^2}
\]
for \( t \in [a, b] \). We obtain

the length of the graph = \( \int_a^b |\gamma'(t)| \, dt = \int_a^b \sqrt{1 + f'(t)^2} \, dt \).

2. Show that
\[
K = \{(x, y) \in \mathbb{R}^2 : x, y, x + y \in [0, 1]\}
\]
is a normal domain (with respect to each coordinate axis).

Note that
\[
K = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1 - x\};
\]

thus, choosing
\[
\phi_1(x) = 0 \quad \text{and} \quad \phi_2(x) = 1 - x
\]
for \( x \in [0, 1] \), we see that \( K \) is a normal domain with respect to the \( x \)-axis. Similarly, one sees that \( K \) is a normal domain with respect to the \( y \) axis.
3. In this problem we compute the hypersurface area \( S_n(r) \) of the \( n \)-sphere

\[
S^n(r) = \{ (x_1, x_2, \ldots x_{n+1}) : x_1^2 + \ldots + x_{n+1}^2 = r^2 \}
\]

by parameterizing the half-sphere \( \Phi = S^n \cap (\mathbb{R}^n \times [0, \infty)) \) as

\[
\Phi(x_1, x_2, \ldots, x_n) = \left( x_1, x_2, \ldots, x_n, \sqrt{r^2 - x_1^2 - x_2^2 - \ldots - x_n^2} \right)
\]

and using the expression

\[
2^n \int_0^r \int_0^{\sqrt{r^2-x_1^2}} \cdots \int_0^{\sqrt{r^2-x_1^2-\ldots-x_n^2}} |N| \, dx_n \ldots dx_2 \, dx_1
\]

for the hypersurface area of \( \Phi \), where

\[
|N| = \left( \frac{1}{n!} \sum_{i_1=1}^{n+1} \sum_{i_2=1}^{n+1} \cdots \sum_{i_n=1}^{n+1} \left| \frac{\partial(\Phi_{i_1}, \Phi_{i_2}, \ldots, \Phi_{i_n})}{\partial(x_1, x_2, \ldots, x_n)} \right|^2 \right)^{1/2}
\]

is the magnitude of the normal vector \( N \) to \( \Phi \).

(a) What are \( S_1(1) \) and \( S_2(1) \)?

\[
S_1(1) = 2\pi, \quad S_2(1) = 4\pi.
\]

(b) Show that \( S_n(r) = r^n S_n(1) \) for \( r > 0 \).

Upon applying the change of variables \( \phi(x) = rx \), where \( x = (x_1, \ldots x_n) \), one immediately obtains from the Jacobian \( J_\phi = r^n \) that \( S_n(r) = r^n S_n(1) \) for \( r > 0 \).

(c) Show that \( |N| \) simplifies to

\[
|N| = \left( 1 + \sum_{j=1}^n \left| \frac{\partial(\Phi_1, \Phi_2, \ldots, \Phi_{j-1}, \Phi_{j+1}, \ldots, \Phi_{n+1})}{\partial(x_1, x_2, \ldots, x_n)} \right|^2 \right)^{1/2}.
\]

The first term arises from omitting \( n + 1 \) from the sequence \((i_1, \ldots, i_n)\); in the remaining terms we have omitted the row corresponding to \( \Phi_j \), for \( j = 1, \ldots n \).
(d) Evaluate \(|N(x_1, \ldots, x_n)|\).

The first \(n - 1\) rows of the determinant in (c) corresponding to index \(j\) are the result of inserting a column of zeros after the \(j - 1\)st column of the \((n - 1) \times (n - 1)\) identity matrix; the last row has the form

\[
\begin{bmatrix}
-\frac{x_1}{\Delta} & -\frac{x_2}{\Delta} & \ldots & -\frac{x_n}{\Delta}
\end{bmatrix},
\]

where \(\Delta = \sqrt{r^2 - x_1^2 - \ldots - x_n^2}\). Here we have used the fact that

\[
\frac{\partial \Phi_{n+1}}{\partial x_j} = -\frac{x_j}{\Delta}.
\]

The determinant of this matrix may be easily found by evaluating along the inserted column of \(n - 1\) zeros; it has absolute value \(|x_j|/\Delta\).

Hence

\[
|N(x_1, \ldots, x_n)| = \left(1 + \sum_{j=1}^{n} \frac{x_j^2}{\Delta^2} \right)^{1/2} = \left(\frac{r^2}{\Delta^2} \right)^{1/2} = \frac{r}{\sqrt{r^2 - x_1^2 - \ldots - x_n^2}}.
\]

(e) For \(n \geq 3\), use part (d) to show that \(S_n(1)\) may be expressed as

\[
S_n(1) = 4 \int_0^1 \int_0^{\sqrt{1-x_1^2}} S_{n-2} \left(\frac{\sqrt{1-x_1^2-x_2^2}}{\sqrt{1-x_1^2-x_2^2}}\right) dx_2 dx_1.
\]

Then compute \(S_n(1)\) in terms of \(S_{n-2}(1)\). Hint: once you have applied part (d) consider changing the variables \((x_1, x_2)\) to a more convenient coordinate system.

We use the fact that \(S_{n-2}(r)\) may be written as

\[
S_{n-2}(r) = 2^{n-1} \int_0^r \int_0^{\sqrt{r^2-x_3^2}} \ldots \int_0^{\sqrt{r^2-x_{n-2}^2}} \frac{r}{\sqrt{r^2 - x_3^2 - \ldots - x_{n-1}^2}} dx_n \ldots dx_4 dx_3
\]
to re-express

\[ S_n(1) = 2^{n+1} \int_0^1 \int_0^1 \int_0^1 \cdots \int_0^1 \frac{1}{\sqrt{1-x_1^2} \cdots - x_n^2} \, dx_n \ldots dx_3 \, dx_2 \, dx_1 \]

\[ = 4 \int_0^1 \int_0^1 S_{n-2}(\sqrt{1-x_1^2-x_2^2}) \, dx_2 \, dx_1 \]

\[ = 4S_{n-2}(1) \int_0^1 \int_0^1 \sqrt{1-x_1^2} (1-x_1^2-x_2^2)^{(n-3)/2} \, dx_2 \, dx_1 \]

\[ = 4S_{n-2}(1) \int_0^{\pi/2} \int_0^1 (1-r^2)^{(n-3)/2} \, r \, dr \, d\theta \]

\[ = 2\pi S_{n-2}(1) \left( \frac{-1}{n-1} \right)^{1} \left( 1-r^2 \right)^{(n-1)/2} \int_0^1 \]

\[ = \frac{2\pi}{n-1} S_{n-2}(1). \]

(f) Use induction to show that

\[
S_n(1) = \begin{cases} 
2\pi^m / (m-1)! & \text{if } n = 2m - 1, \quad m \in \mathbb{N}, \\
(4\pi)^m(m-1)! / (2m-1)! & \text{if } n = 2m, \quad m \in \mathbb{N}.
\end{cases}
\]

If \( S_{2m-1}(1) = 2\pi^m / (m-1)! \),

\[ S_{2m+1}(1) = \frac{2\pi}{2m} \frac{2\pi^m}{(m-1)!} = \frac{2\pi^{m+1}}{m}. \]

Moreover, if \( S_{2m}(1) = (4\pi)^m(m-1)! / (2m-1)! \),

\[ S_{2m+2}(1) = \frac{2\pi}{2m+1} \frac{(4\pi)^m(m-1)!}{(2m-1)!} = \frac{2\pi}{2m+1} \frac{(4\pi)^m2m(m-1)!}{(2m+1)!} = \frac{(4\pi)^{m+1}m!}{(2m+1)!}. \]

Since \( S_1(1) = 2\pi = 2\pi/0! \) and \( S_2(1) = 4\pi = 4\pi0!/1! \), we see by induction that these formulae hold for all \( m \in \mathbb{N} \).

(g) Deduce the astonishing result that \( \lim_{n \to \infty} S_n(1) = 0. \)

This result follows immediately upon application of the ratio test to part (f).

4∗ (a) Let \( a < b \), and let \( f \in C^1([a,b], \mathbb{R}) \) such that \( f \geq 0 \). Viewing the graph of \( f \) as a subset of the \( xy \) plane in \( \mathbb{R}^3 \) and rotating it about the \( x \) axis generates a surface in \( \mathbb{R}^3 \), a so called rotation surface. Show that the area of this surface is

\[ 2\pi \int_a^b f(t) \sqrt{1 + f'(t)^2} \, dt. \]
Let \( r > 0 \), and define

\[
g: [a - r, b + r] \to \mathbb{R}, \quad t \mapsto \begin{cases} f'(a), & t \in [a - r, a], \\ f'(t), & t \in [a, b], \\ f'(b), & t \in [b, b + r]. \end{cases}
\]

Then \( g \) is continuous and extends \( f' \). Define

\[
\tilde{f}: (a - r, b + r) \to \mathbb{R}, \quad t \mapsto f(a) + \int_a^t g(s) \, ds,
\]

so that \( \tilde{f} \) is a \( C^1 \)-function extending \( f \).

The rotation surface can be parametrized as

\[
\Phi(s, t) := s \mathbf{i} + \tilde{f}(s) \cos t \mathbf{j} + \tilde{f}(s) \sin t \mathbf{k},
\]

for \((s, t) \in \mathbb{R} \times (a - r, b + r)\) with parameter domain \([a, b] \times [0, 2\pi]\), so that

\[
\frac{\partial \Phi}{\partial s}(s, t) := \mathbf{i} + f'(s) \cos t \mathbf{j} + f'(s) \sin t \mathbf{k}
\]

and

\[
\frac{\partial \Phi}{\partial t}(s, t) := -f(s) \sin t \mathbf{j} + f(s) \cos t \mathbf{k}
\]

for \( s \in [a, b] \) and \( t \in [0, 2\pi] \). It follows that

\[
N(s, t) = \frac{\partial \Phi}{\partial s}(s, t) \times \frac{\partial \Phi}{\partial t}(s, t)
= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(s) \cos t & f'(s) \sin t \\ 0 & -f(s) \sin t & f(s) \cos t \end{vmatrix}
= f'(s)f(s) \mathbf{i} - f(s) \cos t \mathbf{j} - f(s) \sin t \mathbf{k}
\]

for \( s \in [a, b] \) and \( t \in [0, 2\pi] \) and hence, noting that \( f \) is non-negative,

\[
|N(s, t)| = \sqrt{f(s)^2 f'(s)^2 + f(s)^2 (\sin t)^2 + f(s)^2 (\cos t)^2} = f(s) \sqrt{1 + f'(s)^2}
\]

for \( s \in [a, b] \) and \( t \in [0, 2\pi] \). The definition of surface area and Fubini’s Theorem yield that

\[
\text{surface area} = \int_{[a, b] \times [0, 2\pi]} f(s) \sqrt{1 + f'(s)^2} \\
= \int_0^{2\pi} \left( \int_a^b f(s) \sqrt{1 + f'(s)^2} \, ds \right) \, dt \\
= 2\pi \int_a^b f(s) \sqrt{1 + f'(s)^2} \, ds.
\]
(b) What is the area of the sloped surface of a cone having a circular base with radius $r$ and height $h$?

The outer hull of a cone is the rotation surface obtained from the function

$$f : [0, h] \to \mathbb{R}, \quad t \mapsto \frac{r}{h} t.$$ 

We thus obtain

\[
\text{surface area} = 2\pi \frac{r}{h} \sqrt{1 + \frac{r^2}{h^2}} \int_0^h t \, dt \\
= \pi rh \sqrt{1 + \frac{r^2}{h^2}} \\
= \pi r \sqrt{r^2 + h^2}.
\]