1. An \(N\)-dimensional \textit{cube} is a subset \(C\) of \(\mathbb{R}^N\) such that
\[
C = [x_1 - r, x_1 + r] \times \cdots \times [x_N - r, x_N + r]
\]
with \(x_1, \ldots, x_N \in \mathbb{R}\) and \(r > 0\).

Let \(\emptyset \neq U \subset \mathbb{R}^N\) be open and let \(Z \subset U\) be compact with content zero. Show that, for each \(\epsilon > 0\), there exist cubes \(C_1, \ldots, C_n \subset U\) with
\[
Z \subset C_1 \cup \cdots \cup C_n \quad \text{and} \quad \sum_{j=1}^{n} \mu(C_j) < \epsilon.
\]

2. Let \(D \subset \mathbb{R}^N\) have content. Show that
\[
\mu(D) = \inf \sum_{j=1}^{n} \mu(I_j),
\]
where the infimum on the right-hand side is taken over all \(n \in \mathbb{Z}\) and all compact intervals \(I_1, \ldots, I_n \subset \mathbb{R}^N\) such that \(D \subset I_1 \cup \cdots \cup I_n\).

3. Let \(I \subset \mathbb{R}^N\) be a compact interval, let \(f, g: I \to \mathbb{R}\) be continuous, and let \(\epsilon > 0\). Show that there exists a partition \(P_\epsilon\) of \(I\) such that, for each refinement \(P\) of \(P_\epsilon\), we have
\[
\left| \int_I f g - \sum_{\nu} f(x_\nu)g(y_\nu)\mu(I_\nu) \right| < \epsilon,
\]
where \((I_\nu)_\nu\) is the subdivision of \(I\) corresponding to \(P\) and \(x_\nu, y_\nu \in I_\nu\) are arbitrary.

4. Show that
(a) if \(D \subset \mathbb{R}^N\) has content, then so has \(\overline{D}\), with \(\mu(\overline{D}) = \mu(D)\);
(b) if \(\emptyset \neq U \subset \mathbb{R}^N\) is open, \(Z\) is a set of content zero with \(\overline{Z} \subset U\), and \(\phi: U \to \mathbb{R}^N\) is a \(C^1\)-function, then \(\phi(Z)\) has content zero.
5. (a) Let $D$ be a compact connected subset of $\mathbb{R}^2$ with content $\mu_2(D)$. Suppose we embed $D$ in the plane $z = h$ of $\mathbb{R}^3$. Show that

$$
\mu_3 \doteq \int_{0+}^{h} \int_{\mathbb{R}^2} \chi_D \left( \frac{hx}{z}, \frac{hy}{z} \right) \, dx \, dy \, dz
$$

is the volume of a “cone” formed by projecting $D$ to the origin of $\mathbb{R}^3$, where $\chi_D$ denotes the indicator function over $D$. Compute $\mu_3$ in terms of $\mu_2(D)$. 

Hint: Consider the transformations $(u, v) = \phi_z(x, y) = \left( \frac{zx}{h}, \frac{zy}{h} \right)$ at each fixed $z$.

(b) Generalize part (a) to $n$ dimensions, by projecting a set $D \subset \mathbb{R}^{n-1}$ a (perpendicular) distance $h$ into a new dimension. Compute the content $\mu_n$ of the resulting object.

(c) Describe the objects generated when $n = 1$ and $n = 2$. What are the corresponding formulae for the contents $\mu_1$ and $\mu_2$ of these objects?

6* Let $\emptyset \neq U \subset \mathbb{R}^N$ be open, $K \subset U$ be compact with content, $\phi \in C^1(U, \mathbb{R}^N)$, and suppose that there exists $Z \subset K$ with content zero such that $\det J_\phi(x) \neq 0$ for all $x \in K \setminus Z$. Show that $\phi(K)$ has content. (*Hint: Show that $\partial \phi(K) \subset \phi(Z) \cup \phi(\partial K)$.)