Math 317: Honours Advanced Calculus II  
Winter, 2023     Assignment 2  
January 16, due January 28  
* denotes a bonus question  

1. Let 
\[ f : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto x^2 + y^2. \] 
Show that, there exists \( \epsilon > 0 \) and a \( C^1 \)-function \( \phi : (-\epsilon, \epsilon) \to \mathbb{R} \) with \( \phi(0) = 1 \) such that \( y = \phi(x) \) solves the equation \( f(x, y) = 1 \) for all \( x \in \mathbb{R} \) with \( |x| < \epsilon \). 
Show without explicitly determining \( \phi \) that 
\[ \phi'(x) = -\frac{x}{\phi(x)} \quad (x \in (-\epsilon, \epsilon)). \] 

Let 
\[ g : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto x^2 + y^2 - 1, \] 
so that \( f(x, y) = 1 \) if and only if \( g(x, y) = 0 \). Since 
\[ g(0, 1) = 0 \quad \text{and} \quad \frac{\partial g}{\partial y}(0, 1) = 2, \] 
the existence of \( \epsilon > 0 \) and \( \phi \) follows directly from the Implicit function theorem, with 
\[ \phi'(x) = -\left( \frac{\partial g}{\partial y}(x, \phi(x)) \right)^{-1} \frac{\partial g}{\partial x}(x, \phi(x)) = -\frac{x}{\phi(x)}. \] 

2. (a) Let \( f, g \in C^1(\mathbb{R}^2, \mathbb{R}) \). Given two smooth curves described by \( f(x, y) = 0 \) and \( g(x, y) = 0 \), show that when the distance between points \((\alpha, \beta)\) and \((\xi, \eta)\) lying on the respective curves has an extremum then 
\[ f_y(\alpha, \beta)(\alpha - \xi) = f_x(\alpha, \beta)(\beta - \eta) \] 
and 
\[ g_y(\xi, \eta)(\alpha - \xi) = g_x(\xi, \eta)(\beta - \eta). \] 
We wish to extremize the function 
\[ h(\alpha, \beta, \xi, \eta) = (\alpha - \xi)^2 + (\beta - \eta)^2 \] 
subject to the constraints that \( f(\alpha, \beta) = 0 \) and \( g(\xi, \eta) = 0 \). The Lagrange multiplier theorem tells us to extremize \( h + \lambda f + \mu g \), yielding the equations 
\[ 2(\alpha - \xi) = -\lambda f_x(\alpha, \beta), \] 
\[ 2(\beta - \eta) = -\lambda f_y(\alpha, \beta), \] 
\[ -2(\alpha - \xi) = -\mu g_x(\xi, \eta), \] 
\[ -2(\beta - \eta) = -\mu g_y(\xi, \eta). \]
If \((\alpha, \beta) = (\xi, \eta)\) the desired result holds trivially. Otherwise, \(\lambda \neq 0 \neq \mu\) and the result follows on multiplying the first equation by the second and the third equation by the fourth.

(b) Use part (a) to find the distance between the line \(x + y = 2\) and the ellipse \(x^2 + 2y^2 = 1\).

Let \(f(x, y) = x + y - 2\) and \(g(x, y) = x^2 + 2y^2 - 1\). According to part (a), the extremal distance occurs when

\[
\alpha - \xi = \beta - \eta
\]

and

\[
4\eta(\alpha - \xi) = 2\xi(\beta - \eta).
\]

Since the curves do not intersect, this implies that \(\xi = 2\eta\). Hence \(4\eta^2 + 2\eta^2 = 1\), yielding \(\eta = \pm 1/\sqrt{6}\). Solving the equations \(\alpha + \beta = 2\) and \(\alpha - 2\eta = \beta - \eta\) gives \(\alpha = 1 + \eta/2, \beta = 1 - \eta/2\), so that \(\sqrt{h(\alpha, \beta, \xi, \eta)} = \sqrt{2}(1 - 3\eta/2).\) The minimum distance of \(\sqrt{2} - \sqrt{3}/2\) occurs at \(\eta = 1/\sqrt{6}\). The other point corresponds to a local, but not global, maximum value of \(\sqrt{2} + \sqrt{3}/2\).

3. Determine the minimum and the maximum of

\[
f : \mathbb{R}^3 \to \mathbb{R}, \quad (x, y, z) \mapsto (x - 1)^2 + y^2 + z^2 + 2z
\]

on

\[
K := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 9\}
\]
as well as all points in \(K\) at which they are attained.

We first determine the critical points of \(f\) in the interior of \(K\), i.e., on \(\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 9\}\). As

\[
\frac{\partial f}{\partial x} = 2(x - 1), \quad \frac{\partial f}{\partial y} = 2y, \quad \text{and} \quad \frac{\partial f}{\partial z} = 2(z + 1),
\]

we see that \((1, 0, -1)\) is the only stationary point of \(f\) in the interior of \(K\), with \(f(1, 0, -1) = -1\).

Let

\[
\phi : \mathbb{R}^3 \mapsto \mathbb{R}, \quad (x, y, z) \mapsto z^2 + y^2 + z^2 - 9,
\]

so that \(\partial K = \{(x, y, z) \in \mathbb{R}^3 : \phi(x, y, z) = 0\}\). Let \((x, y, z) \in \partial K\) be such that \(f\) has a local extremum on \(\partial K\), i.e., under the constraint \(\phi(x, y, z) = 0\). The Lagrange multiplier theorem then yields \(\lambda \in \mathbb{R}\) such that \(\nabla f(x, y, z) = \lambda \nabla \phi(x, y, z)\), i.e.,

\[
2x - 2 = \lambda 2x, \quad (2)
\]

\[
2y = \lambda 2y, \quad (3)
\]

\[
2z + 2 = \lambda 2z, \quad (4)
\]
and the constraint yields
\[ x^2 + y^2 + z^2 = 9. \] (5)

**Case 1**: \( y \neq 0 \). In this case, (3) yields \( \lambda = 1 \), so that \( 2x - 2 = 2x \) by (2), which is impossible; so this case cannot occur.

**Case 2**: \( y = 0 \). Adding (2) and (4), we obtain
\[ 2(x + z) = \lambda 2(x + z). \] (6)

**Subcase 2.1**: \( x \neq -z \). Dividing (6) by \( 2(x + z) \) yields \( \lambda = 1 \), and as in Case 1, we see that this is impossible.

**Subcase 2.2**: \( x = -z \). Since \( y = 0 \), (5) yields \( 2x^2 = 9 \), i.e., \( x = \pm \frac{3}{\sqrt{2}} \) and therefore \( z = \mp \frac{3}{\sqrt{2}} \).

We conclude that \( \left( \frac{3}{\sqrt{2}}, 0, -\frac{3}{\sqrt{2}} \right) \) and \( \left( -\frac{3}{\sqrt{2}}, 0, \frac{3}{\sqrt{2}} \right) \) are the only points on \( \partial K \) where \( f \) can attain a local minimum or maximum. The values of \( f \) at these points are
\[ f \left( \frac{3}{\sqrt{2}}, 0, -\frac{3}{\sqrt{2}} \right) = 2 \left( \frac{3 - \sqrt{2}}{\sqrt{2}} \right)^2 - 1 = 10 - 6\sqrt{2} > 1 \]
and
\[ f \left( -\frac{3}{\sqrt{2}}, 0, \frac{3}{\sqrt{2}} \right) = 2 \left( \frac{3 + \sqrt{2}}{\sqrt{2}} \right)^2 - 1 = 10 + 6\sqrt{2} \]

In summary, \( f \) attains its minimum \(-1\) on \( K \) at \((1, 0, -1)\) and its maximum \(10 + 6\sqrt{2}\) at \( \left( -\frac{3}{\sqrt{2}}, 0, \frac{3}{\sqrt{2}} \right) \).

4. (a) Let \( p > 1 \) and \( q > 1 \) be real numbers satisfying \( \frac{1}{p} + \frac{1}{q} = 1 \). Find the minimum value of
\[ f : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto \frac{x^p}{p} + \frac{y^q}{q} \]
subject to the constraints \( xy = 1 \), \( x > 0 \), and \( y > 0 \).

One can use the Lagrange multiplier theorem here, but it is a bit easier in this case just to impose the constraints explicitly. Consider
\[ h : (0, \infty) \to \mathbb{R}, \quad x \mapsto \frac{x^p}{p} + \frac{x^{-q}}{q}. \]

Now \( h'(x) = x^{p-1} - x^{-q-1} \) vanishes only when \( x = 1 \) and \( h''(1) = p + q > 0 \) so \( h \) achieves its minimum value of \( 1/p + 1/q = 1 \) at \( x = 1 \).
(b) Let \( p > 1 \) and \( q > 1 \) be real numbers satisfying \( \frac{1}{p} + \frac{1}{q} = 1 \). If \( a > 0 \) and \( b > 0 \), use part (a) to prove that

\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}.
\]

The result holds trivially if \( a = 0 \) or \( b = 0 \). Otherwise, let \( x = a/(ab)^{1/p} \) and \( y = b/(ab)^{1/q} \). Since \( xy = 1 \), part (a) implies

\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}.
\]

When \( p = q = 2 \), this is the arithmetic–geometric mean inequality for \( \sqrt{a} \) and \( \sqrt{b} \).

(c) Let \( p > 1 \) and \( q > 1 \) be real numbers satisfying \( \frac{1}{p} + \frac{1}{q} = 1 \). If \( a_k \geq 0 \) and \( b_k \geq 0 \), \( k = 1, \ldots, n \), use part (b) to prove

\[
\sum_{k=1}^{n} a_k b_k \leq \left( \sum_{k=1}^{n} a_k^p \right)^{1/p} \left( \sum_{k=1}^{n} b_k^q \right)^{1/q}.
\]

The inequality holds trivially if \( a_k = 0 \) for \( k = 1, \ldots, n \) or \( b_k = 0 \) for \( k = 1, \ldots, n \). Otherwise, let \( A = \left( \sum_{k=1}^{n} a_k^p \right)^{1/p} > 0 \), \( B = \left( \sum_{k=1}^{n} b_k^q \right)^{1/q} > 0 \), \( a = a_k/A \geq 0 \), and \( b = b_k/B \geq 0 \).

Upon multiplying the inequality proved in part(c) by \( AB \) and summing over \( k \) we obtain

\[
\sum_{k=1}^{n} a_k b_k \leq AB \left[ \frac{\sum_{k=1}^{n} (a_k)^p}{p} + \frac{\sum_{k=1}^{n} (b_k)^q}{q} \right] = AB \left( \frac{1}{p} + \frac{1}{q} \right) = \left( \sum_{k=1}^{n} a_k^p \right)^{1/p} \left( \sum_{k=1}^{n} b_k^q \right)^{1/q},
\]

which is known as Hölder’s inequality.

(d) If \( p \geq 1 \) and \( a_k \in \mathbb{R} \) and \( b_k \in \mathbb{R} \) for \( k = 1, \ldots, n \), use part (c) to prove

\[
\left( \sum_{k=1}^{n} |a_k + b_k|^p \right)^{1/p} \leq \left( \sum_{k=1}^{n} |a_k|^p \right)^{1/p} + \left( \sum_{k=1}^{n} |b_k|^p \right)^{1/p}.
\]

If \( p = 1 \) the equality is just the triangle inequality. Otherwise, let \( q = (1 - 1/p)^{-1} \).

Since \( p = 1 + p/q \) we deduce

\[
|a_k + b_k|^p = |a_k + b_k| |a_k + b_k|^{p/q} \leq |a_k| |a_k + b_k|^{p/q} + |b_k| |a_k + b_k|^{p/q}
\]

from the triangle inequality. We then sum this result over \( k \). Since \( q > 1 \) we may apply Hölder’s inequality to each term on the right-hand side:

\[
\sum_{k=1}^{n} |a_k + b_k|^p \leq \left( \sum_{k=1}^{n} |a_k|^p \right)^{1/p} \left( \sum_{k=1}^{n} |a_k + b_k|^p \right)^{1/q} + \left( \sum_{k=1}^{n} |b_k|^p \right)^{1/p} \left( \sum_{k=1}^{n} |a_k + b_k|^p \right)^{1/q}.
\]
Hence we obtain
\[
\left( \sum_{k=1}^{n} |a_k + b_k|^p \right)^{1-1/q} \leq \left( \sum_{k=1}^{n} |a_k|^p \right)^{1/p} + \left( \sum_{k=1}^{n} |b_k|^p \right)^{1/p},
\]
which, since \(1 - 1/q = 1/p\), is the desired result, known as Minkowski’s inequality.

5* Let \(U \subset \mathbb{R}^2\) be open and nonempty, \(f(x, y) : U \to \mathbb{R}^2 \in C^1(U, \mathbb{R}^2)\) such that \(\det J_f = 0\) on \(U\) and \(\frac{\partial f_1}{\partial x}(x_0, y_0) \neq 0\) at some point \((x_0, y_0) \in U\). Prove that there exists a neighbourhood \(V\) of \((x_0, y_0)\) and a function \(\psi : f_1(V) \to \mathbb{R} \in C^1(f_1(V), \mathbb{R})\) such that \(f_2(x, y) = \psi(f_1(x, y))\) for all \((x, y) \in V\).

Denote \(u = f_1(x, y)\) and \(v = f_2(x, y)\). The implicit function theorem guarantees that for each fixed \(u \in f_1(U)\) we may solve \(f_1(x, y) - u = 0\) near \((x_0, y_0)\) uniquely for \(x = \phi_u(y)\), where \(\phi_u \in C^1\). That is, for \(y\) in a neighbourhood of \(y_0\),
\[
\begin{align*}
u &= f_1(\phi_u(y), y), \\
v &= f_2(\phi_u(y), y).
\end{align*}
\]
On differentiating these identities with respect to \(y\), we find in a neighbourhood of \(y_0\) that
\[
\begin{align*}
0 &= \frac{\partial f_1}{\partial x} \frac{\partial \phi_u}{\partial y} + \frac{\partial f_1}{\partial y}, \\
\frac{\partial v}{\partial y} &= \frac{\partial f_2}{\partial x} \frac{\partial \phi_u}{\partial y} + \frac{\partial f_2}{\partial y}.
\end{align*}
\]
Since \(f \in C^1\), we know that \(\frac{\partial f_1}{\partial x} \neq 0\) in a neighbourhood of \((x_0, y_0)\). Thus
\[
\frac{\partial v}{\partial y} = \left( \frac{\partial f_1}{\partial x} \right)^{-1} \left[ \frac{\partial f_2}{\partial x} \frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial x} \frac{\partial f_2}{\partial y} \right] = 0,
\]
on noting that \(\det J_f = 0\) on \(U\). Thus \(v\) is independent of \(y\) in a neighbourhood \(V\) of \((x_0, y_0)\), allowing us to define
\[
\psi(u) : f_1(V) \to \mathbb{R}, \quad u \mapsto f_2(\phi_u(y_0), y_0) = v.
\]
That is, \(f_2(x, y) = \psi(f_1(x, y))\) for all \((x, y) \in V\).

6. Let \((x_0, y_0) \in \mathbb{R}\). Suppose \(f(x, y)\) and \(g(x, y)\) are \(C^1\) functions from \(\mathbb{R}^2 \to \mathbb{R}\), one of which has a nonzero partial derivative at \((x_0, y_0)\). Prove that there exists a function \(\phi \in C^1(\mathbb{R}, \mathbb{R})\) such that \(f(x, y) = \phi(g(x, y))\) or \(g(x, y) = \phi(f(x, y))\) on a neighbourhood of \((x_0, y_0)\) if and only if \(f_x g_y - f_y g_x\) vanishes on a neighbourhood of \((x_0, y_0)\).

\("\Rightarrow\) Suppose \(f(x, y) = \phi(g(x, y))\) on some neighbourhood \(U\) of \((x_0, y_0)\). Then at any point \((x, y) \in U\)
\[
f_x g_y - f_y g_x = \phi'(g)(g_x g_y - g_y g_x) = 0.
\]
The same conclusion holds if \( g(x, y) = \phi(f(x, y)) \).

\( \Leftarrow \) Conversely, suppose that \( f_x g_y - f_y g_x = 0 \) on a neighbourhood \( U \) of \( (x_0, y_0) \). Construct the \( C^1 \) function \( F : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (f(x, y), g(x, y)) \). We are given that \( \det J_F = f_x g_y - f_y g_x = 0 \) on \( U \).

Since \( J_F(x_0, y_0) \) has at least one nonzero element, we can relabel the inputs and outputs of \( F \) so that \( \frac{\partial E_1}{\partial x}(x_0, y_0) \neq 0 \). On a neighbourhood of \( (x_0, y_0) \), we then know from Question 5 that either \( f(x, y) = \phi(g(x, y)) \) or \( g(x, y) = \phi(f(x, y)) \) for some \( C^1 \) function \( \phi \).