1. (a) Let $U \subset \mathbb{R}^N$ be open and convex, and let $f \in C^1(U, \mathbb{R}^N)$ be such that
\[
\det \begin{bmatrix}
\frac{\partial f_1}{\partial x_1}(p_1), & \cdots, & \frac{\partial f_1}{\partial x_N}(p_1) \\
\vdots & \ddots & \vdots \\
\frac{\partial f_N}{\partial x_1}(p_N), & \cdots, & \frac{\partial f_N}{\partial x_N}(p_N)
\end{bmatrix} \neq 0
\]
for all collections of colinear points $(p_1, \ldots, p_N) \in U$. Prove that $f$ is injective. Where is the convexity condition used?

Let $a = (\alpha_1, \ldots, \alpha_N)$ and $b = (\beta_1, \ldots, \beta_N)$ be points in $U$. Suppose that $f(b) = f(a)$. By Taylor’s remainder theorem,
\[
0 = f_i(b) - f_i(a) = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(p_i)(\beta_j - \alpha_j),
\]
for each $i = 1, \ldots, N$, where the points $p_i$ all lie on the line segment joining $a$ and $b$ (which is entirely contained in the convex set $U$). We are given the condition that the determinant of this linear system of equations is nonzero. We may then solve these equations for $\beta_j - \alpha_j$ to deduce that $\beta_j = \alpha_j$ for $j = 1, \ldots, N$.

(b) Prove that $f : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (x^3 - y, e^{x+y})$ is injective.

Since
\[
\begin{vmatrix}
3x_1^2 & -1 \\
e^{x_2+y_2} & e^{x_2+y_2}
\end{vmatrix} = e^{x_2+y_2}(3x_1^2 + 1) > 0
\]
for all points $(x_1, y_1)$ and $(x_2, y_2)$, part (a) implies that $f$ is (globally) injective on $\mathbb{R}^2$.

2. Let $U := \mathbb{R}^2 \setminus \{(0,0)\}$, and let
\[
f : U \to \mathbb{R}^2, \quad (x, y) \mapsto \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right).
\]

(a) Calculate $\det J_f(x, y)$ for all $(x, y) \in U$.

Let $(x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Since
\[
\frac{\partial}{\partial x} \left( \frac{x}{\sqrt{x^2 + y^2}} \right) = \frac{1}{(x^2 + y^2)^{3/2}}[x^2 + y^2 - x^2],
\]
\[
\frac{\partial}{\partial y} \left( \frac{y}{\sqrt{x^2 + y^2}} \right) = \frac{1}{(x^2 + y^2)^{3/2}}[x^2 + y^2 - y^2],
\]
and 
\[ \frac{\partial}{\partial x} \frac{y}{\sqrt{x^2 + y^2}} = \frac{\partial}{\partial y} \frac{x}{\sqrt{x^2 + y^2}} = \frac{1}{(x^2 + y^2)^{3/2}} (-xy), \]

we find 
\[ \det J_f(x, y) = \frac{1}{(x^2 + y^2)^{3/2}} (y^2x^2 - x^2y^2) = 0. \]

(b) Determine \( f(U) \). Does it contain a non-empty open subset of \( \mathbb{R}^2 \)?

As is readily seen (for example, by transforming to polar coordinates), \( f(U) \) is the circle of radius 1 centered at \((0, 0)\). This set does not have interior points and thus contains no nonempty open subset.

3. Is the following “theorem” true?

Let \( U \subset \mathbb{R}^N \) be open and nonempty, let \( x_0 \in U \), and let \( f \in C^1(U, \mathbb{R}^N) \) be such that \( f(V) \) is open for each open neighbourhood \( V \subset U \) of \( x_0 \).

Then \( \det J_f(x_0) \neq 0 \).

Give a proof or provide a counterexample.

No. Let 
\[ f : \mathbb{R} \to \mathbb{R}, \ x \mapsto x^3, \]

and let \( x_0 = 0 \). For any \( x \in \mathbb{R} \) and \( \epsilon > 0 \), we have \( f((x-\epsilon, x+\epsilon)) = ((x-\epsilon)^3, (x+\epsilon)^3) \), so that \( f(V) \) is open for each open subset of \( V \). On the other hand, \( f'(0) = 0 \).

4. Let \( U \subset \mathbb{R}^N \) be open and nonempty, and let \( f \in C^1(U, \mathbb{R}^N) \) be such that \( \det J_f(x) \neq 0 \) for all \( x \in U \).

(a) Show that 
\[ g : U \to \mathbb{R}, \ x \mapsto |f(x)| \]

has no local maximum.

Assume that \( g \) attains a local maximum at \( x_0 \in U \), i.e., there exists an open neighbourhood \( V \subset U \) of \( x_0 \) such that 
\[ |f(x)| \leq |f(x_0)| \quad \forall x \in V. \tag{1} \]

Since \( \det J_f(x) \neq 0 \ \forall x \in U \), it follows that \( f(U) \) is open. Hence, there exists \( \epsilon > 0 \) such that \( B_\epsilon(f(x_0)) \subset f(U) \), contradicting (1).

(b) Suppose that \( U \) is bounded (so that \( \overline{U} \) is compact) and that \( f \) has a continuous extension \( \tilde{f} : \overline{U} \to \mathbb{R}^N \). Show that the continuous map 
\[ \tilde{g} : \overline{U} \to \mathbb{R}, \ x \mapsto |\tilde{f}(x)| \]

attains its maximum on \( \partial U \).

Since \( \overline{U} \) is compact and \( \tilde{f} \) is continuous, \( \tilde{g} \) attains its maximum at a point \( x_0 \in \overline{U} \). By (a), \( x_0 \notin U \); hence \( x_0 \in \partial U \).