Math 225 (Q1) Solution to Homework Assignment 9

1.

(a)

$$T(\underline{e_1}) = T\begin{pmatrix}1\\0\\0\end{pmatrix} = -\underline{b_2} + \underline{b_3}$$

$$T(\underline{e_2}) = T\begin{pmatrix}0\\1\\0\end{pmatrix} = -\underline{b_1} - \underline{b_3}$$

$$T(\underline{e_3}) = T\begin{pmatrix}0\\0\\1\end{pmatrix} = \underline{b_1} - \underline{b_2}.$$

(b)

$$[T(\underline{e_1})]_{\mathcal{B}} = \begin{pmatrix}0\\-1\\1\end{pmatrix}, \quad [T(\underline{e_2})]_{\mathcal{B}} = \begin{pmatrix}-1\\0\\-1\end{pmatrix}, \quad [T(\underline{e_3})]_{\mathcal{B}} = \begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

(c) The matrix for T relative to bases $\mathcal{E}$ and $\mathcal{B}$ is

$$\big[[T(\underline{e_1})]_{\mathcal{B}}\ [T(\underline{e_2})]_{\mathcal{B}}\ [T(\underline{e_3})]_{\mathcal{B}}\big] = \begin{pmatrix}0 & -1 & 1\\ -1 & 0 & -1\\ 1 & -1 & 0\end{pmatrix}.$$

2. The change of coordinates matrix from $\mathcal{B} = \{\underline{b_1}, \underline{b_2}\}$ to $\mathcal{E} = \{\underline{e_1}, \underline{e_2}\}$ is given by

$$P_{\mathcal{E}\leftarrow\mathcal{B}} = [\ [\underline{b_1}]_{\mathcal{E}}\ [\underline{b_2}]_{\mathcal{E}}\ ] = [\ [\underline{b_1}]\ [\underline{b_2}]\ ] = \begin{pmatrix}2 & 1\\ -9 & 8\end{pmatrix}.$$

$$P_{\mathcal{B}\leftarrow\mathcal{E}} = (P_{\mathcal{E}\leftarrow\mathcal{B}})^{-1} = \begin{pmatrix}2 & 1\\ -9 & 8\end{pmatrix}^{-1} = \begin{pmatrix}8/25 & -1/25\\ 9/25 & 2/25\end{pmatrix}.$$

3.

(a) Answer: $\dim(\mathrm{Ran}(T)) = n$. Explanation: Let $\mathcal{B} = \{\underline{b_1}, \cdots, \underline{b_n}\}$ be a basis of V. By the result proved in class, since T is one–one and $\mathcal{B}$ is linearly independent, $T(\mathcal{B}) = \{T(\underline{b_1}, \cdots, T(\underline{b_n})\}$ is also linearly independent. We now show that $T(\mathcal{B})$ is a basis of $\mathrm{Ran}(T)$ so that $\dim(\mathrm{Ran}(T)) = n$. To this end, we only need to verify that $T(\mathcal{B})$ spans $\mathrm{Ran}(T)$. Let $\underline{w} \in \mathrm{Ran}(T)$. Then there exists $\underline{v} \in V$ such that $T(\underline{v}) = \underline{w}$. Since $\mathcal{B}$ spans V, there exist $c_1, \cdots, c_n \in \mathbf{R}$ such that $\underline{v} = c_1\underline{b_1} + \cdots + c_n\underline{b_n}$. Thus,

$$\underline{w} = T(\underline{v}) = T(c_1\underline{b_1} + \cdots + c_n\underline{b_n}) = c_1T(\underline{b_1}) + \cdots + c_nT(\underline{b_n}),$$

so that $\underline{w}$ is a linear combination of vectors in $T(\mathcal{B})$.

(b) Answer: $\dim(\mathrm{Ker}(T)) = n - m$. Explanation: Let $p = \dim(\mathrm{Ker}(T))$. Then $\mathrm{Ker}(T)$ has a basis $\mathcal{B} = \{\underline{b_1}, \cdots, \underline{b_p}\}$. Since $\dim(V) = n$, we can extend $\mathcal{B}$ to a basis $\mathcal{C} = \{\underline{b_1}, \cdots, \underline{b_p}, \underline{b_{p+1}}, \cdots \underline{b_n}\}$ of V. Note that $T(\underline{b_1}) = \cdots = T(\underline{b_p}) = \underline{0}$. We will show that $\{T(\underline{b_{p+1}}), \cdots T(\underline{b_n}\})$ is a basis of W so that $m = n - (p+1) + 1 = n - p$. Thus, $\dim(\mathrm{Ker}(T)) = p = n - m$.

First, we will show that $\{T(\underline{b_{p+1}}), \cdots T(\underline{b_n})\}$ is linearly independent. Suppose

$$x_{p+1}T(\underline{b_{p+1}}) + \cdots + x_n T(\underline{b_n}) = \underline{0}, \qquad \text{for some} \quad x_{p+1}, \cdots x_n \in \mathbf{R}.$$

We want to show that $x_{p+1} = \cdots = x_n = 0$. Now, since T is linear, we have, $T(x_{p+1}\underline{b_{p+1}} + \cdots + x_n\underline{b_n}) = \underline{0}$. This shows that $\underline{z} = x_{p+1}\underline{b_{p+1}} + \cdots + x_n\underline{b_n} \in \mathrm{Ker}(T)$. Since $\{\underline{b_1}, \cdots, \underline{b_p}\}$ spans $\mathrm{Ker}(T)$, there exist $x_1, \cdots, x_p \in \mathbf{R}$ such that

$$x_{p+1}\underline{b_{p+1}} + \cdots + x_n\underline{b_n} = \underline{z} = x_1\underline{b_1} + \cdots + x_p\underline{b_n}.$$

Thus,

$$(-x_1)\underline{b_1} + \cdots + (-x_p)\underline{b_p} + x_{p+1}\underline{b_{p+1}} + \cdots + x_n\underline{b_n} = \underline{0}.$$

Since $\{\underline{b_1}, \cdots, \underline{b_n}\}$ is linearly independent, $(-x_1) = \cdots = (-x_p) = x_{p+1} = \cdots = x_n = 0$. In particular, $x_{p+1} = \cdots = x_n = 0$.

Next, we will show that $\{T(\underline{b_{p+1}}), \cdots T(\underline{b_n})\}$ spans W. In class, we showed that since $\{\underline{b_1}, \cdots, \underline{b_p}, \underline{b_{p+1}}, \cdots \underline{b_n}\}$ spans V and T is onto, therefore $\{T(\underline{b_1}), \cdots, T(\underline{b_p}), T(\underline{b_{p+1}}), \cdots, T(\underline{b_n})\}$ spans W. Now $T(\underline{b_1}) = \cdots = T(\underline{b_p}) = \underline{0}$, therefore $\{T(\underline{b_{p+1}}), \cdots, T(\underline{b_n})\}$ spans W.

4.

(a)

$$< f, g >= \int_{-1}^{1} f(x)g(x)\,dx = \int_{-1}^{1} (x^2 - x)(x-1)\,dx = -\frac{4}{3},$$

$$||f|| = \sqrt{< f, f >} = \sqrt{\int_{-1}^{1} (x^2 - x)^2, dx} = \frac{4}{\sqrt{15}}$$

and

$$||g|| = \sqrt{< g, g >} = \sqrt{\int_{-1}^{1} (x-1)^2\,dx} = \frac{\sqrt{8}}{\sqrt{3}}.$$

(b) $\cos\theta = \frac{<f,g>}{||f||\ ||g||} = -\frac{\sqrt{5}}{\sqrt{8}}$.

(c) $||f-g|| = \sqrt{< f-g, f-g >} = \sqrt{\int_{-1}^{1}(x^2-2x+1)^2\,dx} = \frac{4\sqrt{2}}{\sqrt{5}}$.

(d)

$$\hat{f}(x) = f(x) = x^2 - x$$

$$\hat{g}(x) = g - \frac{< g, f >}{< f, f >} f = (x-1) - \frac{-4/3}{16/15}(x^2-x) = \frac{5}{4}x^2 - \frac{1}{4}x - 1.$$

(e) The best mean square approximation of h by function in W is the projection $\hat{h}$ of h onto the subspace W. Therefore,

$$\hat{h} = \frac{< h, \hat{f} >}{< \hat{f}, \hat{f} >}\hat{f} + \frac{< h, \hat{g} >}{< \hat{g}, \hat{g} >}\hat{g} = \frac{2/5}{16/15}\hat{f} + \frac{-1/6}{1}\hat{g}$$

so that

$$\hat{h}(x) = \frac{3}{8}(x^2-x) - \frac{1}{6}(\frac{5}{4}x^2 - \frac{1}{4}x - 1) = \frac{1}{6}x^2 - \frac{1}{3}x + \frac{1}{6}.$$

5.

(a) Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ and let $\alpha, \beta \in \mathbf{R}$. Then

$$\begin{aligned} T(\alpha A + \beta B) &= T\begin{pmatrix} \alpha a_{11} + \beta b_{11} & \alpha a_{12} + \beta b_{12} \\ \alpha a_{21} + \beta b_{21} & \alpha a_{22} + \beta b_{22} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -(\alpha a_{12} + \beta b_{12}) \\ \alpha a_{21} + \beta b_{21} & \alpha a_{11} + \beta b_{11} \end{pmatrix} \\ &= \alpha\begin{pmatrix} 0 & -a_{12} \\ a_{21} & a_{11} \end{pmatrix} + \beta\begin{pmatrix} 0 & -b_{12} \\ b_{21} & b_{11} \end{pmatrix} = \alpha T(A) + \beta T(B). \end{aligned}$$

(b)

$$\begin{aligned} A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Ker}(T) &\Longleftrightarrow T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Longleftrightarrow -b = c = a = 0 \\ &\Longleftrightarrow A = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}. \end{aligned}$$

So $\mathrm{Ker}(T) = \left\{ d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ : \ d \in \mathbf{R} \right\}$ and $\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a basis of $\mathrm{Ker}(T)$.

(c)

$$B \in \mathrm{Ran}(T) \Longleftrightarrow B = \begin{pmatrix} 0 & -b \\ c & a \end{pmatrix} = -b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, $\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ spans $\mathrm{Ran}(T)$. Since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are clearly linearly independent (none of the matrices is a linearly combinatin of the other two), therefore $\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a basis of $\mathrm{Ran}(T)$.