Math 225 (Q1) Homework Assignment 4.

1. Let \( u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ u_2 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} \) and \( u_3 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \).
   
   (a) Show that \( \{u_1, u_2, u_3\} \) is an orthogonal set.
   
   (b) Using part (a), express \( x = \begin{pmatrix} 8 \\ -2 \\ 3 \end{pmatrix} \) as a linear combination of \( u_1, u_2 \) and \( u_3 \).

2. Let \( S \subset \mathbb{R}^n \) be a non–empty subset of \( \mathbb{R}^n \). Define
   
   \[ S^\perp = \{x \in \mathbb{R}^n : x \cdot y = 0, \text{ for all } y \in S\}. \]
   
   Show that \( S^\perp \) is a subspace of \( \mathbb{R}^n \), that is, \( S^\perp \) is non-empty and for all vectors \( u, v \in S^\perp \) and for all scalars \( \alpha, \beta \in \mathbb{R} \), we have \( \alpha u + \beta v \in S^\perp \). \( S^\perp \) is called the “orthogonal complement” of \( S \).

3. Let \( A = \begin{pmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{pmatrix} \). Using the Gram–Schmidt process, find an orthonormal basis for the column space of \( A \). (Recall that the column space of \( A \), \( \text{Col}(A) \), is the span of the column vectors of \( A \)).

4. Let \( A \) be a \( m \times n \) matrix, where \( m \) and \( n \) may not be the same,
   
   (a) Show that if the columns of \( A \) are linearly dependent, then there exists a non–zero vector \( x \in \mathbb{R}^n \) such that \( Ax = 0 \).
   
   (b) Show that if \( A^T A \) is invertible, then the columns of \( A \) are linearly independent. Hint: Use Question 4 in Assignment 1.

5. (a) A square matrix \( U \) is said to be an “orthogonal” matrix if \( U \) is invertible and \( U^{-1} = U^T \). Let \( U, V \) be two \( n \times n \) orthogonal matrices. Show that \( UV \) is also an orthogonal matrix.
   
   (b) Suppose \( A = PRP^{-1} \), where \( A, P, R \) are \( n \times n \) matrices, \( P \) is an orthogonal matrix and \( R \) is an upper triangular matrix. This is called a “Schur” factorization/decomposition of \( A \). (Note that this says in particular that the matrix \( A \) is
similar to the matrix \( R \). Show that if \( A \) is symmetric, then \( R \) is also symmetric and hence \( R \) is a diagonal matrix.