Math 225 (Q2) Homework Assignment 3.

1. Let $A = \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}$. A is called a "stochastic matrix", meaning that, the entries of A are non–negative and sum of the entries in each column of A is 1. Let $\underline{v_1} = \begin{pmatrix} \frac{3}{7} \\ \frac{4}{7} \end{pmatrix}$ and $\underline{x_0} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$.

(a) Find a "basis" for $\mathbf{R^2}$ consisting of $\underline{v_1}$ and another eigenvector $\underline{v_2}$ of A.

(b) Verify that $\underline{x_0}$ can be written in the form $\underline{x_0} = \underline{v_1} + c\underline{v_2}$. Find c

(c) For $k = 1, 2, \cdots$, define $\underline{x_k} = A\underline{x_{k-1}}$, that is, $\underline{x_k} = A^k \underline{x_0}$. Compute $\underline{x_1}$ and $\underline{x_2}$. Write down a formula for $\underline{x_k}$, in terms of k, $\underline{v_1}$, $\underline{v_2}$ and the eigenvalues oi A. Show that $\underline{x_k}$ tends to $\underline{v_1}$ as k gets larger and larger, that is, $\lim_{k\to\infty} \underline{x_k} = \underline{v_1}$. (This is called the "power method" for finding eigenvectors numerically).

2. Consider the complex numbers $z = 2 - 3i$ and $w = 3 + 4i$. Express the following in the form of $a + bi$, where a, b are real numbers.

$$\text{(a)}\quad z\overline{z}, \qquad \text{(b)}\quad |z|, \qquad \text{(c)}\quad zw, \qquad \text{(d)}\quad \frac{z}{w},$$

where $\overline{z}$ is the complex conjugate of z and $|z|$ is the modulus of z.

3. Let $A = \begin{pmatrix} 0 & 1 \\ -8 & 4 \end{pmatrix}$. We think of A as a complex matrix.

(a) Find the (complex) eigenvalues of A. Also find a basis for each eigenspace of A in $\mathbf{C}^2$.

(b) Diagonalize A. That is, find a (complex) diagonal matrix D and a (complex) invertible matrix P such that $P^{-1}AP = D$.

(c) Find a real matrix C of the form $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and a real invertible matrix Q such that that $Q^{-1}AQ = C$.

4. Let A be a $n \times n$ real, symmetric matrix, that is, $A^T = A$. Let $\underline{x} \in \mathbf{C^n}$ be a complex vector and let $q = \overline{\underline{x}}^T A\underline{x}$. Show that q is a real number. Hint: Justify the following chain of equalities

$$\overline{q} = \overline{\overline{\underline{x}}^T A\underline{x}} = \underline{x}^T\ \overline{A\underline{x}} = \underline{x}^T A\overline{\underline{x}} = (\underline{x}^T A\overline{\underline{x}})^T = \overline{\underline{x}}^T A^T \underline{x} = q.$$

5. Prove the parallelogram law: Let $\underline{u}, \underline{v} \in \mathbf{R}^n$. Then

$$||\underline{u} + \underline{v}||^2 + ||\underline{u} - \underline{v}||^2 = 2||\underline{u}||^2 + 2||\underline{v}||^2.$$