Math 225 (Q2) Homework Assignment 2.

1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) Use quadratic formula and show that the eigenvalues of A are:

$$\lambda = \frac{1}{2}\left[(a+d) \pm \sqrt{(a-d)^2 + 4bc}\right].$$

(b) Let $D = (a-d)^2 + 4bc$. D is called the discriminant. Show that A has two distinct real eigenvalues if $D > 0$.

(c) Show that A has one repeated real eigenvalue if $D = 0$.

(d) Show that A has no real eigenvalue if $D < 0$.

2. Let $A = \begin{pmatrix} 3 & 4 & -1 \\ -1 & -2 & 1 \\ 3 & 9 & 0 \end{pmatrix}$.

(a) Find the characteristic equation, $\det(A - \lambda I) = 0$, of A.

(b) Find the eigenvalues of A. For each eigenvalue of A, state its algebraic multiplicity.

(c) For each eigenvalue of A, state its geometric multiplicity and find a basis for the corresponding eigenspace.

(d) Show that A is not diagonliazable.

3. Let A be a $n \times n$ matrix and suppose A has n real eigenvalues: $\lambda_1, \cdots, \lambda_n$ (the $\lambda_i's$ may not be all distinct).

(a) Show that the characteristic polynomial, $p_A(\lambda) := \det(A - \lambda I)$, of A can be expressed as

$$p_A(\lambda) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda).$$

(b) Using part (a), show that $\det(A) = \lambda_1 \cdots \lambda_n$, that is, $\det(A)$ is the product of the eigenvalues of A.

4. Let $A = \begin{pmatrix} 7 & 4 \\ -3 & -1 \end{pmatrix}$

(a) Find the characteristic polynomial, $p_A(\lambda) := \det(A - \lambda I)$, of A.

(b) Find the eigenvalues and eigenvectors of A.

(c) Diagonalize A.

(d) Find A^{10} using part (c).

(e) Using part (c), find a matrix B such that $B^2 = A$ (B is called a "square root" of A).

5. Let A be a square matrix.

(a) Show that A and A^T have the same eigenvalues.

(b) Show that 0 is an eigenvalue of A if and only if A is not invertible (also called "singular", that is $\det A = 0$).