Math 225 (Q1) Solution to Homework Assignment 1.

1. (a)

$$\text{LHS} = \underline{u} \cdot \underline{v} = u_1 v_1 + \cdots + u_n v_n.$$

$$\text{RHS} = \underline{u}^T \underline{v} = (u_1, \cdots, u_n) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1 v_1 + \cdots + u_n v_n.$$

Since LHS (left hand side) and RHS (right hand side) are the same, the proof is complete.

(b) Note that $||\underline{u}|| = \sqrt{\underline{u} \cdot \underline{u}} = \sqrt{u_1^2 + \cdots + u_n^2}$. Thus,

$$||\underline{u}|| = 0 \Longleftrightarrow ||\underline{u}||^2 = 0 \Longleftrightarrow u_1^2 + \cdots + u_n^2 = 0$$

$$\Longleftrightarrow u_1^2 = \cdots = u_n^2 = 0, \quad \text{since } u_1^2 \geq 0,\ u_2^2 \geq 0, \text{ etc.}$$

$$\Longleftrightarrow u_1 = \cdots = u_n = 0 \Longleftrightarrow \underline{u} = \underline{0}.$$

2. The (i,j)-th entry of the matrix on the LHS can be expressed as

$$[(AB)^T]_{i.j} = [AB]_{j,i} = \sum_k A_{j,k} B_{k,i} = \sum_k B_{k,i} A_{j,k} = \sum_k [B^T]_{i,k} [A^T]_{k,j} = [B^T A^T]_{i,j}$$

which is the (i,j)-th entry of the matrix on the RHS.

3. (a) To show that $\text{Ker}(T)$ is a subspace of the domain $\mathbf{R}^n$, we need to establish the following three claims.

Claim 1: $\underline{0} \in \text{Ker}(T)$.

Proof. $T(\underline{0}) = T(0\ \underline{0}) = 0\ T(\underline{0}) = \underline{0}$.

Claim 2: $\underline{u}, \underline{v} \in \text{Ker}(T)$ implies $\underline{u} + \underline{v} \in \text{Ker}(T)$.

Proof. Let $\underline{u}, \underline{v} \in \text{Ker}(T)$. Then $T(\underline{u}) = \underline{0}$ and $T(\underline{v}) = \underline{0}$. Now,

$$T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v}) = \underline{0} + \underline{0} = \underline{0}$$

so that $\underline{u} + \underline{v} \in \text{Ker}(T)$.

Claim 3: $c \in \mathbf{R}$ and $\underline{u} \in \text{Ker}(T)$ imply $c\underline{u} \in \text{Ker}(T)$.

Proof. Let $\underline{u} \in \text{Ker}(T)$. Then $T(\underline{u} = \underline{0}$. Now,

$$T(c\underline{u}) = c\ T(\underline{u}) = c\ \underline{0} = \underline{0}$$

so that $c\,\underline{u} \in \mathrm{Ker}(T)$, as desired.

(b) To show that $\mathrm{Ran}(T)$ is a subspace of the codomain $\mathbf{R}^m$, we need to establish the following two claims.

Claim 1: $\underline{0} \in \mathrm{Ran}(T)$.

Proof. We proved $T(\underline{0}) = \underline{0}$ in part (a). Thus, $\underline{0} \in \mathrm{Ran}(T)$.

Claim 2: $c, d \in \mathbf{R}$ and $\underline{u}, \underline{v} \in \mathrm{Ran}(T)$ imply $c\,\underline{u} + d\,\underline{v} \in \mathrm{Ran}(T)$.

Proof. Let $\underline{u}, \underline{v} \in \mathrm{Ran}(T)$. Then there exist $\underline{x}, \underline{y} \in \mathbf{R}^n$ such that $T(\underline{x}) = \underline{u}$ and $T(\underline{y}) = \underline{v}$. Now $c\underline{x} + d\underline{y} \in \mathbf{R}^n$ and

$$T(c\underline{x} + d\underline{y}) = c\,T(\underline{x}) + d\,T(\underline{y}) = c\,\underline{u} + d\,\underline{v}$$

so that $c\,\underline{u} + d\,\underline{v} \in \mathrm{Ran}(T)$, as desired.

4. Claim 1: If $\underline{x} \in \mathrm{Nul}(A)$, then $\underline{x} \in \mathrm{Nul}(A^T A)$.

Proof. Let $\underline{x} \in \mathrm{Nul}(A)$. Then $A\underline{x} = \underline{0}$. Multiply by A^T on the left, we get,

$$(A^T A)\underline{x} = A^T(A\underline{x}) = A^T\underline{0} = \underline{0}.$$

Thus, $\underline{x} \in \mathrm{Nul}(A^T A)$.

Claim 2: If $\underline{x} \in \mathrm{Nul}(A^T A)$, then $\underline{x} \in \mathrm{Nul}(A)$.

Proof. Let $\underline{x} \in \mathrm{Nul}(A^T A)$. Then $A^T A\underline{x} = (A^T A)\underline{x} = \underline{0}$. Now

$$||A\underline{x}||^2 = (A\underline{x}) \cdot (A\underline{x}) = (A\underline{x})^T(A\underline{x}) = \underline{x}^T A^T A\underline{x} = \underline{x}^T\underline{0} = 0$$

so that $||A\underline{x}|| = 0$. Thus, $A\underline{x} = \underline{0}$ and hence $\underline{x} \in \mathrm{Nul}(A)$.

5.
$$\begin{aligned}\underline{p} = \underline{u} - \underline{v} &= (\underline{u} - \underline{w}) + (\underline{w} - \underline{v}) = -(\underline{w} - \underline{u}) - (\underline{v} - \underline{w}) \\ &= -\underline{r} - \underline{q} = (-1)\underline{q} + (-1)\underline{r}\end{aligned}$$

so that $\underline{p}$ is a linear combination of $\underline{q}$ and $\underline{r}$. Thus, $\underline{p}$, $\underline{q}$, $\underline{r}$ are linear dependent.