Math 225 (Q1) Solution to Homework Assignment 1.

1. (a) 
\[ \text{LHS} = u \cdot v = u_1v_1 + \cdots + u_nv_n. \]
\[ \text{RHS} = u^T v = (u_1, \cdots, u_n) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1v_1 + \cdots + u_nv_n. \]
Since LHS (left hand side) and RHS (right hand side) are the same, the proof is complete.

(b) Note that \( ||u|| = \sqrt{u \cdot u} = \sqrt{u_1^2 + \cdots + u_n^2}. \) Thus,
\[ ||u|| = 0 \iff ||u||^2 = 0 \iff u_1^2 + \cdots + u_n^2 = 0 \]
\[ \iff u_1 = \cdots = u_n = 0, \text{ since } u_1^2 \geq 0, u_2^2 \geq 0, \text{ etc.} \]
\[ \iff u_1 = \cdots = u_n = 0 \iff u = 0. \]

2. The \((i, j)\)-th entry of the matrix on the LHS can be expressed as
\[ [(AB)^T]_{i,j} = [AB]_{j,i} = \sum_k A_{j,k}B_{k,i} = \sum_k B_{k,i}A_{j,k} = \sum_k [B^T]_{i,k}[A^T]_{k,j} = [B^T A^T]_{i,j} \]
which is the \((i, j)\)-th entry of the matrix on the RHS.

3. (a) To show that \( \ker(T) \) is a subspace of the domain \( \mathbb{R}^n \), we need to establish the following three claims.

Claim 1: \( 0 \in \ker(T). \)
Proof. \( T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0. \)

Claim 2: \( u, v \in \ker(T) \) implies \( u + v \in \ker(T). \)
Proof. Let \( u, v \in \ker(T). \) Then \( T(u) = 0 \) and \( T(v) = 0. \) Now,
\[ T(u + v) = T(u) + T(v) = 0 + 0 = 0 \]
so that \( u + v \in \ker(T). \)

Claim 3: \( c \in \mathbb{R} \) and \( u \in \ker(T) \) imply \( cu \in \ker(T). \)
Proof. Let \( u \in \ker(T). \) Then \( T(u) = 0. \) Now,
\[ T(cu) = c T(u) = c 0 = 0 \]
so that \( c \, u \in \text{Ker}(T) \), as desired.

(b) To show that \( \text{Ran}(T) \) is a subspace of the codomain \( \mathbb{R}^m \), we need to establish the following two claims.

Claim 1: \( 0 \in \text{Ran}(T) \).

Proof. We proved \( T(0) = 0 \) in part (a). Thus, \( 0 \in \text{Ran}(T) \).

Claim 2: \( c, d \in \mathbb{R} \) and \( u, v \in \text{Ran}(T) \) imply \( c \, u + d \, v \in \text{Ran}(T) \).

Proof. Let \( u, v \in \text{Ran}(T) \). Then there exist \( x, y \in \mathbb{R}^n \) such that \( T(x) = u \) and \( T(y) = v \). Now \( cx + dy \in \mathbb{R}^n \) and

\[
T(cx + dy) = c \, T(x) + d \, T(y) = c \, u + d \, v
\]

so that \( c \, u + d \, v \in \text{Ran}(T) \), as desired.

4. Claim 1: If \( x \in \text{Nul}(A) \), then \( x \in \text{Nul}(A^T A) \).

Proof. Let \( x \in \text{Nul}(A) \). Then \( A^T x = 0 \). Multiply by \( A^T \) on the left, we get,

\[
(A^T A)x = A^T (A^T x) = A^T 0 = 0.
\]

Thus, \( x \in \text{Nul}(A^T A) \).

Claim 2: If \( x \in \text{Nul}(A^T A) \), then \( x \in \text{Nul}(A) \).

Proof. Let \( x \in \text{Nul}(A^T A) \). Then \( A^T Ax = (A^T A)x = 0 \). Now

\[
||A^T x||^2 = (Ax) \cdot (Ax) = (Ax)^T (Ax) = x^T A^T Ax = x^T 0 = 0
\]

so that \( ||A^T x|| = 0 \). Thus, \( A^T x = 0 \) and hence \( x \in \text{Nul}(A) \).

5. \[
p = u - v = (u - w) + (w - v) = -(w - u) - (v - w)
= -x - q = (-1)q + (-1)r
\]

so that \( p \) is a linear combination of \( q \) and \( r \). Thus, \( p, q, r \) are linear dependent.