Math 225 (Q1) Homework Assignment 1.

1. Let \( u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \ v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \) be two vectors in \( \mathbb{R}^n \). Recall, the dot product of \( u \) and \( v \) is defined as
\[
    u \cdot v = u_1 v_1 + \cdots + u_n v_n.
\]
and the length (also called magnitude or norm) of \( u \) is defined as
\[
    ||u|| = \sqrt{u \cdot u}.
\]
(a) Show that
\[
    u \cdot v = u^T v,
\]
where \( u^T = (u_1, \ldots, u_n) \) is the transpose of \( u \).
(b) Show that \( ||u|| = 0 \) if and only if \( u = 0 \), that is, the vector \( u \) has zero length if and only if \( u \) is the zero vector.

2. Let \( A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \) be a \( m \times n \) matrix and let \( B = (b_{jk})_{1 \leq j \leq n, 1 \leq k \leq p} \) be a \( n \times p \) matrix. Recall that the product \( C \) of the matrices \( A \) and \( B \) is a \( m \times p \) matrix, \( C = (c_{ik})_{1 \leq i \leq m, 1 \leq k \leq p} \), where the \((i, k)\)-th entry of \( C \) is given by
\[
    c_{ik} = a_{i1} b_{1k} + a_{i2} b_{2k} + \cdots + a_{in} b_{nk}.
\]
Note that \( c_{ik} \) is the “dot product” of the \( i \)-th row of \( A \) with the \( k \)-th column of \( B \). Also recall that the transpose of \( A \) is the matrix \( A^T \) which is a \( n \times m \) matrix whose \( i \)-th row is the \( i \)-th column of \( A \). That is, if \( D = A^T \) and \( D = (d_{rs})_{1 \leq r \leq n, 1 \leq s \leq m} \), then
\[
    d_{rs} = a_{sr}, \quad \text{for all } 1 \leq r \leq n, 1 \leq s \leq m.
\]
Show that \( (AB)^T = B^T A^T \).

3. Let \( T : \mathbb{R}^n \to \mathbb{R}^m \) be a linear transformation, that is, \( T \) is a mapping taking vectors in \( \mathbb{R}^n \) to vectors in \( \mathbb{R}^m \) and \( T \) satisfies the properties
\[
    T(u + v) = T(u) + T(v)
\]
\[
    T(cu) = cT(u)
\]
for all \( u, v \in \mathbb{R}^n \) and all \( c \in \mathbb{R} \).
for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Recall, the kernel of $T$ is the subset of $\mathbb{R}^n$ defined by

$$\text{Ker}(T) = \{ \mathbf{u} \in \mathbb{R}^n : T(\mathbf{u}) = 0 \}.$$ 

and the range of $T$ is the subset of $\mathbb{R}^m$ defined by

$$\text{Ran}(T) = \{ \mathbf{v} \in \mathbb{R}^m : \mathbf{v} = T(\mathbf{u}) \text{ for some } \mathbf{u} \in \mathbb{R}^n \}.$$ 

(a) Show that $\text{Ker}(T)$ is a subspace of $\mathbb{R}^n$.

(b) Show that $\text{Ran}(T)$ is a subspace of $\mathbb{R}^m$.

4. Let $A$ be a $m \times n$ matrix. Recall that the null space of $A$ is the set

$$\text{Nul}(A) = \{ \mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0} \}.$$ 

Show that $\text{Nul}(A^T A) = \text{Nul}(A)$.

5. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be any three vectors in $\mathbb{R}^n$. Define the vectors $\mathbf{p} = \mathbf{u} - \mathbf{v}$, $\mathbf{q} = \mathbf{v} - \mathbf{w}$ and $\mathbf{r} = \mathbf{w} - \mathbf{u}$. Show that $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are linearly dependent by expressing one of the vectors as a linear combination of the other two vectors.