1. Compute $\tan\left(\frac{\pi}{4} + \frac{1}{40}\right)$ correctly rounded to 2 places after the decimal point using Taylor’s Remainder Theorem. Use the Lagrange form of the remainder to establish whether you should round your result up or down.

Let $f(x) = \tan x$, $a = \pi/4$, and $b = a + 1/40$. Taylor’s theorem states that

$$f(b) = f(a) + (b-a)f'(a) + R_2,$$

where $R_2 = \frac{1}{2}(b-a)^2 f''(\gamma)$ and $a < \gamma < b$. Hence

$$\tan b = \tan a + \left(\frac{1}{40}\right) \frac{1}{\cos^2 a} + R_2 = 1 + \frac{1}{20} + R_2 = 1.05 + R_2,$$

with

$$R_2 = \left(\frac{1}{40}\right)^2 \frac{\sin \gamma}{\cos^3 \gamma}.$$  

We have $a < \gamma < b \leq \pi/3$ since $\pi/12 \geq 1/40$. Hence $\cos \gamma > 1/2$ and $0 < \sin \gamma < 1$, so

$$0 < R_2 < \left(\frac{1}{1600}\right) \frac{1}{(1/2)^3} = \frac{1}{200}.$$

Since $R_2 < 0.005$, we conclude that $\tan\left(\frac{\pi}{4} + \frac{1}{40}\right) \approx 1.05$, correctly rounded to 2 places after the decimal point.

2. Let $f(x, y) = x^2y^2 + 3xy + y$.

(a) Find and classify all the stationary points of $f$.

The equations

$$0 = \frac{\partial f}{\partial x} = 2xy^2 + 3y = (2xy + 3)y,$$

$$0 = \frac{\partial f}{\partial y} = 2x^2y + 3x + 1 = x(2xy + 3) + 1,$$

are simultaneously satisfied only when $(x, y) = (-1/3, 0)$. At this point, the Hessian

$$\begin{pmatrix}
2y^2 & 4xy + 3 \\
4xy + 3 & 2x^2
\end{pmatrix}$$

evaluates to

$$\begin{pmatrix}
0 & 3 \\
3 & \frac{2}{3}
\end{pmatrix},$$

with determinant $-9 < 0$. The Hessian is thus indefinite; we conclude that $f$ has a saddle point at $(-1/3, 0)$. 

(b) In what direction does $f$ change the most at the point $(1/2, -3)$?
In the direction of $\nabla f(1/2, -3) = (0, 1)$ (i.e. in the $y$-direction).

3. Find and classify all the stationary points of $f(x, y) = x^3 + 3x^2y + 12y^2 - 36y + 1$.

$$0 = \frac{\partial f}{\partial x} = 3x^2 + 6xy,$$
$$0 = \frac{\partial f}{\partial y} = 3x^2 + 24y - 36,$$
leading to the stationary points $(0, 3/2), (6, -3), (2, 1)$.
The Hessian
$$\begin{pmatrix} 6x + 6y & 6x \\ 6x & 24 \end{pmatrix} = 6 \begin{pmatrix} x + y & x \\ x & 4 \end{pmatrix}$$
has positive determinant at $(0, 3/2)$ with positive diagonal entries, so this point corresponds to a minimum. The points $(6, -3), (2, 1)$ are saddle points because the determinant is negative there.

4. Let $x, y \in \mathbb{R}$. Show that there is $\theta \in (0, 1)$ such that

$$\sin(x + y) = x + y - \frac{1}{2}(x^2 + 2xy + y^2) \sin(\theta(x + y)).$$

Let
$$f: \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto \sin(x + y).$$

By Taylor’s theorem, there is $\theta \in (0, 1)$, such that

$$f(x, y) = f(0, 0) + (\text{grad } f)(0, 0) \cdot (x, y) + \frac{1}{2}(\text{Hess } f)(\theta x, \theta y)(x, y) \cdot (x, y).$$

Clearly, $f(0, 0) = 0$. Since

$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial y}(x, y) = \cos(x + y),$$

we have

$$(\text{grad } f)(0, 0) \cdot (x, y) = (1, 1) \cdot (x, y) = x + y.$$
Moreover, since
\[
\frac{\partial^2 f}{\partial x^2}(x, y) = \frac{\partial^2 f}{\partial y^2}(x, y) = \frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y) = -\sin(x + y)
\]
we have
\[
(Hess \ f)(\theta x, \theta y)(x, y) \cdot (x, y) = \left(\begin{array}{c} -\sin(\theta(x + y)) - \sin(\theta(x + y)) \\ -\sin(\theta(x + y)) - \sin(\theta(x + y)) \end{array}\right) \cdot \left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{c} -\sin(\theta(x + y))(x + y) \\ -\sin(\theta(x + y))(x + y) \end{array}\right) \cdot \left(\begin{array}{c} x \\ y \end{array}\right) = -(x^2 + 2xy + y^2) \sin(\theta(x + y)).
\]
Hence,
\[
\sin(x + y) = x + y - \frac{1}{2}(x^2 + 2xy + y^2) \sin(\theta(x + y))
\]
does hold as claimed.

5. An \(N \times N\) matrix \(X\) is invertible if there is \(X^{-1} \in M_N(\mathbb{R})\) such that \(XX^{-1} = X^{-1}X = I_N\).

(a) Show that \(U \doteq \{X \in M_N(\mathbb{R}) : X \text{ is invertible}\}\) is open. (Hint: \(X \in M_N(\mathbb{R})\) is invertible if and only if \(\det X \neq 0\).)
Since \(\det : M_N(\mathbb{R}) \to \mathbb{R}\) is continuous and \(\mathbb{R} \setminus \{0\}\) is open, \(U = \det^{-1}(\mathbb{R} \setminus \{0\})\) is open.

(b) Show that
\[
f : U \to M_N(\mathbb{R}), \quad X \mapsto X^{-1}
\]
is totally differentiable on \(U\) and calculate \(Df(X_0)\) for each \(X_0 \in U\). (Hint: By Cramer’s rule, \(f\) is continuous.)
Let \(X_0 \in U\). Since \(U\) is open by (i), \(X_0 + H \in U\) for \(|H|\) sufficiently small. Note that
\[
(X_0 + H)^{-1} - X_0^{-1} = -X_0^{-1}((X_0 + H) - X_0)(X_0 + H)^{-1} = -X_0^{-1}H(X_0 + H)^{-1}.
\]
Define
\[
T : M_N(\mathbb{R}) \to M_N(\mathbb{R}), \quad X \mapsto -X_0^{-1}XX_0^{-1}.
\]
For $|H|$ sufficiently small, we have

\[
\frac{|f(X_0 + H) - f(X_0) - TH|}{|H|} = \frac{1}{|H|} \left| X_0^{-1} H (X_0 + H)^{-1} - X_0^{-1} H X_0^{-1} \right| \\
= \left| X_0^{-1} \frac{H}{|H|} \left( (X_0 + H)^{-1} - X_0^{-1} \right) \right|.
\]

As $|H| \rightarrow 0$, the term $X_0^{-1} \frac{H}{|H|}$ stays bounded whereas $(X_0 + H)^{-1} - X_0^{-1} \rightarrow 0$ by the continuity of $f$. It follows that

\[
\lim_{|H| \rightarrow 0} \frac{|f(X_0 + H) - f(X_0) - TH|}{|H|} = 0.
\]

Hence, $f$ is differentiable at $X_0$ and $Df(X_0) = T$.

6. Let $\emptyset \neq U \subset \mathbb{R}^N$ be open, and let $f : U \rightarrow \mathbb{R}$ be partially differentiable such that $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N}$ are bounded. Show that $f$ is continuous.

Let $x \in U$, and choose $\epsilon > 0$ such that $B_{\epsilon}(x) \subset U$. Let $\xi \in \mathbb{R}^N$ with $|\xi| < \epsilon$, so that $x + \xi \in B_{\epsilon}(x)$. For $j = 0, 1, \ldots, N$, let

\[
x^{(j)} = x + \sum_{\nu=1}^{j} \xi_{\nu} e_{\nu},
\]

so that $x^{(0)} = x$ and $x^{(N)} = x + \xi$. For $j = 1, \ldots, N$, define

\[
g^{(j)} : [0, 1] \rightarrow \mathbb{R}, \quad t \mapsto f(x^{(j-1)} + t \xi_j e_j),
\]

so that $g^{(j)}(0) = f(x^{(j-1)})$ and $g^{(j)}(1) = f(x^{(j)})$. By the mean value theorem in one variable, there is $\theta_j \in (0, 1)$ such that

\[
f(x^{(j)}) - f(x^{(j-1)}) = g^{(j)}(1) - g^{(j)}(0) = \frac{d}{dt} g^{(j)}(\theta_j) = \frac{\partial f}{\partial x_j} (x^{(j-1)} + \theta_j \xi_j e_j) \xi_j.
\]

Let $C \geq 0$ be such that

\[
\left| \frac{\partial f}{\partial x_j}(x) \right| \leq C \quad (j = 1, \ldots, N, x \in U).
\]
We then obtain:

\[
|f(x + \xi) - f(x)| = \left| \sum_{j=1}^{N} f(x^{(j)}) - f(x^{(j-1)}) \right|
\]

\[
\leq \sum_{j=1}^{N} |f(x^{(j)}) - f(x^{(j-1)})|
\]

\[
= \sum_{j=1}^{N} \left| \frac{\partial f}{\partial x_j}(x^{(j-1)}) + \theta_j \xi_j e_j \xi_j \right|
\]

\[
\leq C \sum_{j=1}^{N} |\xi_j|.
\]

It is clear that the right hand side of this inequality tends to zero as \(\xi \to 0\). Hence, the left hand side does the same, i.e. \(f\) is continuous at \(x\).

7. Let \(c_1, \ldots, c_p \in \mathbb{R}^N\). For which \(x \in \mathbb{R}^N\) does \(\sum_{j=1}^{p} |x - c_j|^2\) become minimal?

Let

\[
f : \mathbb{R}^N \to \mathbb{R}, \quad (x_1, \ldots, x_N) \mapsto \sum_{j=1}^{p} \sum_{k=1}^{N} (x_k - c_{j,k})^2
\]

Then for \(k = 1, \ldots, N\),

\[
\frac{\partial f}{\partial x_k}(x_1, \ldots, x_N) = \sum_{j=1}^{p} 2(x_k - c_{j,k}).
\]

Hence, \(x = (x_1, \ldots, x_N) \in \mathbb{R}^N\) is a zero of the gradient of \(f\) if and only if

\[
x_k = \frac{1}{p} \sum_{j=1}^{p} c_{j,k} \quad (k = 1, \ldots, N),
\]

i.e.

\[
x = \frac{1}{p} \sum_{j=1}^{p} c_j.
\]

Since

\[
(Hess \ f)(x) = \begin{bmatrix}
2 & 0 & \cdots & 0 \\
0 & 2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 2
\end{bmatrix}
\]

is positive definite for all \(x \in \mathbb{R}^N\), it follows that \(f\) attains a local minimum (which is also a global minimum because there are no other stationary points) at \(x = \frac{1}{p} \sum_{j=1}^{p} c_j\).
8. Does \( \mathbb{Q} \cap [0, 1] \) have (Jordan) content zero?

No. Assume that there are compact intervals \( I_1, \ldots, I_n \subset [0, 1] \) such that

\[
\mathbb{Q} \cap [0, 1] \subset \bigcup_{j=1}^{n} I_j \quad \text{and} \quad \sum_{j=1}^{n} \mu(I_j) < \frac{1}{2}.
\]

This latter condition implies that \( \bigcup_{j=1}^{n} I_j \neq [0, 1] \). Since \( I_1, \ldots, I_n \) are closed, \( U := \mathbb{R} \setminus \bigcup_{j=1}^{n} I_j \) is open. Since \( 0, 1 \in \mathbb{Q} \cap [0, 1] \subset \bigcup_{j=1}^{n} I_j \), it follows that \( U \cap (0, 1) \neq \emptyset \).

Choose \( x_0 \in U \cap (0, 1) \). Choose \( \epsilon > 0 \) such that \( (x_0 - \epsilon, x_0 + \epsilon) \subset U \cap (0, 1) \). Then \( (x_0 - \epsilon, x_0 + \epsilon) \) contains at least one rational number, which is a contradiction.