Lemma 5.1 (Partition Refinement): If $P$ and $Q$ are partitions of $[a,b]$ such that $Q \supset P$, then

$$\mathcal{L}(P, f) \leq \mathcal{L}(Q, f) \leq \mathcal{U}(Q, f) \leq \mathcal{U}(P, f).$$

Lemma 5.2 (Upper Sums Bound Lower Sums): Let $f$ be bounded on $[a,b]$. If $P$ and $Q$ are any partitions of $[a,b]$, then

$$\mathcal{L}(P, f) \leq \mathcal{U}(Q, f).$$

Lemma 5.3 (Lower Integrals vs. Upper Integrals): Let $f$ be bounded on $[a,b]$. Then

$$\int_a^b f \leq \int_a^b f.$$

Theorem 5.1 (Integrability): $\int_a^b f$ exists and equals $\alpha \iff$ there exists a sequence of partitions $\{P_n\}_{n=1}^{\infty}$ of $[a,b]$ such that

$$\lim_{n \to \infty} \mathcal{L}(P_n, f) = \alpha = \lim_{n \to \infty} \mathcal{U}(P_n, f).$$

Theorem 5.2 (Cauchy Criterion for Integrability): Suppose $f$ is bounded on $[a,b]$. Then $\int_a^b f$ exists $\iff$ for each $\epsilon > 0$ there exists a partition $P$ of $[a,b]$ such that

$$U(P, f) - L(P, f) < \epsilon.$$

Corollary 5.2.1 (Piecewise Integration): Suppose $a < c < b$. Then

$$\int_a^b f \in \iff \int_a^c f \in \text{ and } \int_c^b f \in.$$

Furthermore, when either side holds,

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Theorem 5.3 (Darboux Integrability Theorem): $\int_a^b f$ exists and equals $\alpha \iff$ for any sequence of partitions $P_n$ having subinterval widths that go to zero as $n \to \infty$, all Riemann sums $\mathcal{S}(P_n, f)$ converge to $\alpha$.  

Theorem 5.4 (Linearity of Integral Operator): Suppose \( \int_a^b f \) and \( \int_a^b g \) exist. Then

(i) \( \int_a^b (f + g) = \int_a^b f + \int_a^b g \)

(ii) \( \int_a^b (cf) = c \int_a^b f \) for any constant \( c \in \mathbb{R} \).

Theorem 5.5 (Integral Bounds): Suppose

(i) \( \int_a^b f \exists \),

(ii) \( m \leq f(x) \leq M \) for \( x \in [a, b] \).

Then

\[ m(b - a) \leq \int_a^b f \leq M(b - a). \]

Corollary 5.5.1 (Preservation of Non-Negativity): If \( f(x) \geq 0 \) for all \( x \in [a, b] \) and \( \int_a^b f \) exists then \( \int_a^b f \geq 0 \).

Corollary 5.5.2 (Continuity of Integrals): Suppose \( \int_a^b f \) exists. Then the function \( F(x) = \int_a^x f \) is continuous on \( [a, b] \).

Theorem 5.6 (Integrability of Continuous Functions): If \( f \) is continuous on \( [a, b] \) then \( \int_a^b f \) exists.

Theorem 5.7 (Integrability of Monotonic Functions): If \( f \) is monotonic on \( [a, b] \) then \( \int_a^b f \) exists.

Lemma 5.4 (Families of Antiderivatives): Let \( F_0(x) \) be an antiderivative of \( f \) on an interval \( I \). Then \( F \) is an antiderivative of \( f \) on \( I \) \iff \( F(x) = F_0(x) + C \) for some constant \( C \).

Theorem 5.8 (Antiderivatives at Points of Continuity): Suppose

(i) \( \int_a^b f \) exists;

(ii) \( f \) is continuous at \( c \in (a, b) \).

Then \( f \) has the antiderivative \( F(x) = \int_a^x f \) at \( x = c \).

Corollary 5.8.1 (Antiderivative of Continuous Functions): If \( f \) is continuous on \( [a, b] \) then \( f \) has an antiderivative on \( [a, b] \).

Theorem 5.9 (Fundamental Theorem of Calculus [FTC]): Let \( f \) be integrable and have an antiderivative \( F \) on \( [a, b] \). Then

\[ \int_a^b f = F(b) - F(a). \]
Corollary 5.9.1 (FTC for Continuous Functions): Let $f$ be continuous on $[a, b]$ and let $F$ be any antiderivative of $f$ on $[a, b]$. Then
\[
\int_a^b f = F(b) - F(a).
\]

Theorem 5.10 (Mean Value Theorem for Integrals): Suppose $f$ is continuous on $[a, b]$. Then
\[
\int_a^b f = f(c)(b - a)
\]
for some number $c \in [a, b]$.

Theorem 7.1 (Change of Variables): Suppose $g'$ is continuous on $[a, b]$ and $f$ is continuous on $g([a, b])$. Then
\[
\int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du.
\]

Theorem 7.2 (Integration by Parts): Suppose $f'$ and $g'$ are continuous functions on $[a, b]$. Then
\[
\int_a^b fg' = [fg]_a^b - \int_a^b f'g.
\]

Lemma 7.1 (Polynomial Factors): If $z_0$ is a root of a polynomial $P(z)$ then $P(z)$ is divisible by $(z - z_0)$.

Lemma 7.2 (Linear Partial Fractions): Suppose that $P(x)/Q(x)$ is a proper rational function such that $Q(x) = (x - a)^nQ_0(x)$, where $Q_0(a) \neq 0$ and $n \in \mathbb{N}$. Then there exists a constant $A$ and a polynomial $P_0$ with $\deg P_0 < \deg Q - 1$ such that
\[
\frac{P(x)}{Q(x)} = \frac{A}{(x - a)^n} + \frac{P_0(x)}{(x - a)^{n-1}Q_0(x)}.
\]

Lemma 7.3 (Quadratic Partial Fractions): Let $x^2 + \gamma x + \lambda$ be an irreducible quadratic polynomial (i.e. $\gamma^2 - 4\lambda < 0$). Suppose that $P(x)/Q(x)$ is a proper rational function such that $Q(x) = (x^2 + \gamma x + \lambda)^mQ_0(x)$, where $Q_0(x)$ is not divisible by $(x^2 + \gamma x + \lambda)$ and $m \in \mathbb{N}$. Then there exists constants $\Gamma$ and $\Lambda$ and a polynomial $P_0$ with $\deg P_0 < \deg Q - 2$ such that
\[
\frac{P(x)}{Q(x)} = \frac{\Gamma x + \Lambda}{(x^2 + \gamma x + \lambda)^m} + \frac{P_0(x)}{(x^2 + \gamma x + \lambda)^{m-1}Q_0(x)}.
\]
Theorem 7.3 (Linear Interpolation Error): Let $f$ be a twice-differentiable function on $[0, h]$ satisfying $|f''(x)| \leq M$ for all $x \in [0, h]$. Let

$$L(x) = f(0) + \frac{f(h) - f(0)}{h} x.$$ 

Then

$$\int_0^h |L(x) - f(x)| \, dx \leq \frac{M h^3}{12}.$$ 

Corollary 7.3.1 (Trapezoidal Rule Error): Let $P$ be a uniform partition of $[a, b]$ into $n$ subintervals of width $h = (b - a)/n$, and $f$ be a twice-differentiable function on $[a, b]$ satisfying $|f''(x)| \leq M$ for all $x \in [a, b]$. Then the error $E_n^T = T_n - \int_a^b f$ of the uniform Trapezoidal Rule

$$T_n = h \sum_{i=1}^n \frac{f(x_{i-1}) + f(x_i)}{2},$$ 

satisfies

$$|E_n^T| \leq \frac{n M h^3}{12} = \frac{M(b-a)^3}{12n^2}.$$ 

Theorem 8.1 (Pappus' Theorems): Let $\mathcal{L}$ be a line in a plane.

(i) If a curve lying entirely on one side of $\mathcal{L}$ is rotated about $\mathcal{L}$, the area of the surface generated is the product of the length of the curve times the distance travelled by the centroid.

(ii) If a region lying entirely on one side of $\mathcal{L}$ is rotated about $\mathcal{L}$, the volume of the solid generated is the product of the area of the region times the distance travelled by the centroid.

Theorem 9.1 (Increasing Functions: Bounded $\iff$ Asymptotic Limit Exists): Let $f$ be a monotonic increasing function on $[a, \infty)$. Then $f$ is bounded on $[a, \infty) \iff \lim_{x \to \infty} f$ exists.

Corollary 9.1.1 (Improper Integrals of Non-Negative Functions): Let $f$ be a non-negative function that is integrable on $[a, T]$ for all $T \geq a$. If there exists a bound $B$ such that $\int_a^T f \leq B$ for all $T \geq a$, then $\int_a^\infty f$ converges.

Corollary 9.1.2 (Comparison Test): Suppose $0 \leq f(x) \leq g(x)$ and $\int_a^T f$ and $\int_a^T g$ exist for all $T \geq a$. Then

(i) $\int_a^\infty g \in \mathcal{C} \Rightarrow \int_a^\infty f \in \mathcal{C};$

(ii) $\int_a^\infty f \in \mathcal{D} \Rightarrow \int_a^\infty g \in \mathcal{D}$.
Corollary 9.1.3 (Limit Comparison Test): Let $f$ and $g$ be positive integrable functions satisfying
\[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = L. \]

(i) For $0 < L < \infty$ we have $\int_a^\infty g \in \mathcal{C} \iff \int_a^\infty f \in \mathcal{C}$.

(ii) When $L = 0$ we can only say $\int_a^\infty g \in \mathcal{C} \Rightarrow \int_a^\infty f \in \mathcal{C}$.

Theorem 9.2 (Cauchy Criterion for Improper Integrals): Let $f$ be a function.

(i) Suppose $\int_a^t f$ exists for all $t \in (a, b)$. Then $\int_a^b f \in \mathcal{C} \iff \forall \epsilon > 0, \exists \delta > 0$
\[ x, y \in (b - \delta, b) \Rightarrow \left| \int_x^y f \right| < \epsilon; \]

(ii) Suppose $\int_a^T f$ exists for all $T > a$. Then $\int_a^\infty f \in \mathcal{C} \iff \forall \epsilon > 0, \exists T$ such that
\[ T_2 \geq T_1 \geq T \Rightarrow \left| \int_{T_1}^{T_2} f \right| < \epsilon. \]

Theorem 9.3 (Cauchy Criterion for Infinite Series): The infinite series $\sum_{k=1}^\infty a_k$ converges if and only if for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that
\[ m > n \geq N \Rightarrow \left| \sum_{k=n}^{m} a_k \right| < \epsilon. \]

Theorem 9.4 (Divergence Test): If $\sum_{k=1}^\infty a_k \in \mathcal{C}$ then $\lim_{n \to \infty} a_n = 0$.

Theorem 9.5 (Non-Negative Terms: Convergence $\iff$ Bounded Partial Sums): If $a_k \geq 0$ and $S_n = \sum_{k=1}^n a_k$ then $\sum_{k=1}^\infty a_k \in \mathcal{C} \iff \{S_n\}_{n=1}^\infty$ is a bounded sequence.

Corollary 9.5.1 (Comparison Test): If $0 \leq a_k \leq b_k$ for $k \in \mathbb{N}$ then

(i) $\sum_{k=1}^\infty b_k \in \mathcal{C} \Rightarrow \sum_{k=1}^\infty a_k \in \mathcal{C}$;

(ii) $\sum_{k=1}^\infty a_k \in \mathcal{D} \Rightarrow \sum_{k=1}^\infty b_k \in \mathcal{D}$. 

5
Corollary 9.5.2 (Limit Comparison Test): Suppose $a_k \geq 0$ and $b_k > 0$ for $k \in \mathbb{N}$ and $\lim_{k \to \infty} a_k/b_k = L$. Then

(i) if $0 < L < \infty$: $\sum_{k=1}^{\infty} a_k \in \mathcal{C} \iff \sum_{k=1}^{\infty} b_k \in \mathcal{C}$;

(ii) if $L = 0$: $\sum_{k=1}^{\infty} b_k \in \mathcal{C} \Rightarrow \sum_{k=1}^{\infty} a_k \in \mathcal{C}$.

Corollary 9.5.3 (Ratio Comparison Test): If $a_k > 0$ and $b_k > 0$ and

$$\frac{a_{k+1}}{a_k} \leq \frac{b_{k+1}}{b_k}$$

for all $k \geq N$, then

(i) $\sum_{k=1}^{\infty} b_k \in \mathcal{C} \Rightarrow \sum_{k=1}^{\infty} a_k \in \mathcal{C}$;

(ii) $\sum_{k=1}^{\infty} a_k \in \mathcal{D} \Rightarrow \sum_{k=1}^{\infty} b_k \in \mathcal{D}$.

Corollary 9.5.4 (Ratio Test): Suppose $a_k > 0$ and $b_k > 0$.

(i) If $\exists$ a number $x < 1$ such that $\frac{a_{k+1}}{a_k} \leq x$ for all $k \geq N$, then $\sum_{k=1}^{\infty} a_k \in \mathcal{C}$.

(ii) If $\exists$ a number $x \geq 1$ such that $\frac{a_{k+1}}{a_k} \geq x$ for all $k \geq N$, then $\sum_{k=1}^{\infty} a_k \in \mathcal{D}$.

Corollary 9.5.5 (Limit Ratio Test): Suppose $a_k > 0$ for all $k \in \mathbb{N}$ and

$$\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = c.$$  

Then

(i) $0 \leq c < 1 \Rightarrow \sum_{k=1}^{\infty} a_k \in \mathcal{C}$,

(ii) $c > 1 \Rightarrow \sum_{k=1}^{\infty} a_k \in \mathcal{D}$,

(iii) $c = 1 \Rightarrow$?
Theorem 9.6 (Integral Test): Suppose $f$ is continuous, decreasing, and non-negative on $[1, \infty)$. Then
\[
\sum_{k=1}^{\infty} f(k) \in \mathcal{C} \iff \int_{1}^{\infty} f \in \mathcal{C}.
\]

Theorem 9.7 (Absolute Convergence): An absolutely convergent series is convergent.

Theorem 9.8 (Radius of Convergence): For each power series $\sum_{k=0}^{\infty} c_k x^k$ there exists a number $R$, called the radius of convergence, with $0 \leq R \leq \infty$, such that
\[
\sum_{k=0}^{\infty} c_k x^k \in \begin{cases} 
\text{Abs} \mathcal{C} & \text{if } |x| < R, \\
\mathcal{D} & \text{if } |x| > R, \\
? & \text{if } |x| = R.
\end{cases}
\]

Lemma A.1 (Complex Conjugate Roots): Let $P$ be a polynomial with real coefficients. If $z$ is a root of $P$, then so is $\overline{z}$.

Theorem A.1 (Fundamental Theorem of Algebra): Any non-constant polynomial $P(z)$ with complex coefficients has a complex root.

Corollary A.1.1 (Polynomial Factorization): Every complex polynomial $P(z)$ of degree $n \geq 0$ has exactly $n$ complex roots $z_1, z_2, \ldots, z_n$ and can be factorized as $P(z) = A(z - z_1)(z - z_2)\ldots(z - z_n)$, where $A \in \mathbb{C}$.

Corollary A.1.2 (Real Polynomial Factorization): Every polynomial with real coefficients can be factorized as
\[
P(x) = A(x - a_1)^{n_1}\ldots(x - a_k)^{n_k}(x^2 + \gamma_1 x + \lambda_1)^{m_1}\ldots(x^2 + \gamma_\ell x + \lambda_\ell)^{m_\ell}.
\]