1. Consider the function \( f(x) = \frac{1}{1 + x^2} \) on \([0, 1]\).

   (a) Construct a uniform partition \( P \) on \([0, 1]\) with 2 subintervals of equal width.

   \[
P = \left\{0, \frac{1}{2}, 1\right\}.
   \]

   (b) Compute the lower sum \( L(P, f) \).

   Since the partition is uniform,

   \[
   L(P, f) = \frac{1}{2} \left( \frac{4}{5} + \frac{1}{2} \right) = \frac{13}{20}.
   \]

   (c) Compute the upper sum \( U(P, f) \).

   \[
   U(P, f) = \frac{1}{2} \left( 1 + \frac{4}{5} \right) = \frac{9}{10}.
   \]

   (d) Use your results from part (b) and (c) to prove that

   \[
   \frac{13}{5} \leq \pi \leq \frac{18}{5}.
   \]

   We see that

   \[
   \frac{13}{20} = L(P, f) \leq \int_0^1 \frac{1}{1 + x^2} \, dx = \tan^{-1} 1 = \frac{\pi}{4} \leq U(P, f) = \frac{9}{10},
   \]

   Thus \( \frac{13}{5} \leq \pi \leq \frac{18}{5} \). Note that the average of these bounds, 31/10, is quite close to the exact value of \( \pi \).
2. (a) Let

\[ I_n = \int \sec^n x \, dx. \]

For \( n \geq 2 \), prove that

\[ I_n = \frac{\sec^{n-1} x \sin x}{n-1} + \frac{n-2}{n-1} I_{n-2}. \]

Hint: express \( \sec^n x = \sec^{n-2} x \cdot \sec^2 x \).

\[ I_n = \int \sec^{n-2} x \sec^2 x \, dx \]

\[ = \sec^{n-2} x \tan x - (n-2) \int \sec^{n-3} x \sec x \tan x \tan x \, dx \]

\[ = \sec^{n-1} x \sin x - (n-2) \int \sec^{n-3} x \sec x (\sec^2 x - 1) \, dx \]

\[ = \sec^{n-1} x \sin x - (n-2)(I_n - I_{n-2}). \]

On solving for \( n \) we obtain the desired result.

(b) Use part (a) to find

\[ \int \sec^3 x \, dx. \]

On setting \( n = 3 \) we find from part (a) that

\[ \int \sec^3 x \, dx = \frac{\sec^2 x \sin x}{2} + \frac{1}{2} \int \sec x \, dx = \frac{1}{2} (\sec x \tan x + \log |\sec x + \tan x|) + C. \]

(c) Use part (b) to find

\[ \int_0^1 \sqrt{1 + u^2} \, du. \]

On letting \( u = \tan x \), we find

\[ \int_0^1 \sqrt{1 + u^2} \, du = \int_0^{\pi/4} \sec^3 x \, dx = \frac{1}{2} \left[ \sqrt{2} + \log \left( 1 + \sqrt{2} \right) \right]. \]
3. Find

\[ I = \int \frac{u + 1}{u(u^2 + 9)} \, du. \]

Express

\[ \frac{u + 1}{u(u^2 + 9)} = \frac{A}{u} + \frac{Bu + C}{u^2 + 9} \]

and equate coefficients of like powers in

\[ u + 1 = A(u^2 + 9) + (Bu + C)u \]

to obtain the system of equations

\[ u^2 : 0 = A + B, \]
\[ u^1 : 1 = C, \]
\[ u^0 : 1 = 9A, \]

which has the unique solution \( A = 1/9, B = -1/9, C = 1. \)

Thus

\[ I = \frac{1}{9} \log |u| - \frac{1}{18} \log (u^2 + 9) + \frac{1}{3} \arctan \frac{u}{3} + K, \]

where \( K \) is an arbitrary constant.

4. (a) Let \( f \) be a strictly increasing positive differentiable function on some interval \([a, b]\), with inverse \( f^{-1}. \)

Prove that

\[ \int_{f(a)}^{f(b)} \frac{f^{-1}(y)}{y} \, dy = [x \log f(x)]_a^b - \int_{a}^{b} \log f(x) \, dx. \]

Change the integration variable on the left-hand side to \( x = f^{-1}(y) \), so that \( y = f(x) \)
and \( dy = f'(x) \, dx \). Then integrate by parts:

\[ \int_{f(a)}^{f(b)} \frac{f^{-1}(y)}{y} \, dy = \int_{a}^{b} x \frac{f'(x)}{f(x)} \, dx = [x \log f(x)]_a^b - \int_{a}^{b} \log f(x) \, dx. \]

(b) Use part (a) to find an antiderivative for \( \log x. \)

On setting \( y = f(x) = x \), so that \( f^{-1}(y) = y \), an indefinite version of part (a) leads to

\[ \int 1 \, dx = x \log x - \int \log x \, dx. \]

Hence \( \int \log x \, dx = x \log x - x + C. \)