1. Compute

\[ \int \frac{1}{(12 - 8x - 4x^2)^{3/2}} \, dx. \]

On letting \( u = x + 1 \) and then \( u = 2 \sin x \) we obtain

\[ \int \frac{1}{8(3 - 2x - x^2)^{3/2}} \, dx = \frac{1}{8} \int \frac{1}{(4 - u^2)^{3/2}} \, dx = \frac{1}{8} \int \frac{1}{2 \cos^3 x} \, 2 \cos x \, dx \]
\[ = \frac{1}{32} \int \frac{1}{\cos^2 x} \, dx = \frac{1}{32} \tan x + C = \frac{u}{32 \sqrt{4 - u^2}} + C = \frac{x + 1}{32 \sqrt{3 - 2x - x^2}} + C. \]

2. Let \( f \) be a function with a continuous \((n + 1)\)st derivative on \([a, b]\) for some \( n \in \mathbb{N} \). Consider the function \( g(t) = (b - t)^n \).

(a) Find \( g'(t) \).

\[ -n(b - t)^{n-1}. \]

(b) Find the \( k \)th derivative \( g^{(k)}(t) \) of \( g \) for all \( k \in \mathbb{N} \). What happens for \( k > n \)?

\[
\begin{cases} 
(-1)^k \frac{n!}{(n-k)!} (b-t)^{n-k} & \text{for } 1 \leq k < n, \\
(-1)^n n! & \text{for } k = n, \\
0 & \text{for } k > n.
\end{cases}
\]

(c) Show that \( g^{(n)}(b) = (-1)^n n! \) and that \( g^{(k)}(b) = 0 \) for all \( k \neq n \).

This follows directly on substituting \( t = b \) in part (b).

(d) By multiplying \( f \) by \( g^{(n+1)} \) and integrating, show that

\[ \int_a^b f g^{(n+1)} = 0. \]

This follows from the fact that \( g^{(n+1)} = 0 \).
(e) Use induction on \(m\) to show for \(m = 1, 2, \ldots, n, n + 1\) that
\[
\int f g^{(n+1)} = \sum_{k=0}^{m-1} (-1)^k f^{(k)} g^{(n-k)} + (-1)^m \int f^{(m)} g^{(n+1-m)}.
\]
The case \(m = 1\) is just the integration by parts formula:
\[
\int f g^{(n+1)} = f^{(0)} g^{(n)} - \int f^{(1)} g^{(n)}.
\]
Moreover, if the statement holds for the case \(m\) then an additional integration by parts in the final term leads to
\[
\int f g^{(n+1)} = \sum_{k=0}^{m-1} (-1)^k f^{(k)} g^{(n-k)} + (-1)^m \int f^{(m+1)} g^{(n-m)}
\]
\[
= \sum_{k=0}^{m} (-1)^k f^{(k)} g^{(n-k)} + (-1)^{m+1} \int f^{(m+1)} g^{(n-m)},
\]
which is precisely the desired result with \(m\) replaced by \(m + 1\).
We can continue this induction process only until \(m = n + 1\), when the \(g\) in the integrand of the desired result is differentiated \(0\) times.

(f) Consider now the implication of parts (d) and (e) for \(m = n + 1\), when the indefinite integrals are evaluated between \(a\) and \(b\):
\[
0 = \left[ \sum_{k=0}^{n} (-1)^k f^{(k)} g^{(n-k)} \right]_a^b + (-1)^n \int_a^b f^{(n+1)} g^{(0)}.
\]
Use this result to obtain a version of Taylor’s theorem where the remainder is expressed in integral form:
\[
f(b) = \sum_{k=0}^{n} \frac{(b-a)^k}{k!} f^{(k)}(a) + \frac{1}{n!} \int_a^b (b-t)^n f^{(n+1)}(t) \, dt.
\]
We find from parts (b) and (c) that
\[
0 = f^{(0)}(b)(-1)^n n! - \sum_{k=0}^{n} (-1)^k f^{(k)}(a)(-1)^{n-k} \frac{n!}{k!} (b-a)^k - (-1)^n \int_a^b f^{(n+1)} g^{(0)}.
\]
On dividing by \((-1)^n n!\) we obtain the desired result.

(g) Our derivation extends to the case \(n = 0\) simply by setting \(g(t) = 1\) on \([a, b]\).
What theorem does the result in part (f) reduce to when \(n = 0\)?
This is just the statement of the Fundamental Theorem of Calculus:
\[
f(b) = f(a) + \int_a^b f'(t) \, dt.
\]
3. (a) Find the area bounded by the curves \( y = e^x, \ y = ex, \) and \( x = 0. \)

First note that \( y = e^x \) and \( y = ex \) intersect at \( x = 1. \) For \( x \) in \([0, 1),\) the difference of the two curves, \( e^x - ex, \) is decreasing and for \( x \) in \((1, \infty),\) the difference is increasing. Hence \( e^x \geq ex \) for all real \( x,\) with equality holding only at \( x = 1.\) The area is thus

\[
\int_0^1 (e^x - ex) \, dx = \left[ e^x - \frac{e^{x^2}}{2} \right]_0^1 = e - e\frac{1}{2} - 1 = \frac{e}{2} - 1.
\]

(b) A cable hanging between two poles located at \( x = -b \) and \( x = b \) has the shape of a catenary, \( y = a \cosh \frac{x}{a}. \) Compute the length of the cable.

\[
\int_{-b}^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_{-b}^b \sqrt{1 + \sinh^2 \frac{x}{a}} \, dx = \int_{-b}^b \cosh \frac{x}{a} \, dx = \left[ a \sinh \frac{x}{a} \right]_{-b}^b = 2a \sinh \frac{b}{a}.
\]

4. The area swept out by a radial vector from the origin to a point \((\cos t, \sin t)\) on a unit circle is easily seen to be

\[
\left( \frac{t}{2\pi} \right) \pi = \frac{t}{2}.
\]

(a) Verify this result for \( t \in [0, \pi/2)\) by computing the area bounded by the line \( y = x \tan t, \) the positive \( x \) axis, and the unit circle \( x^2 + y^2 = 1,\)

Given a point \((x, y) = (\cos s, \sin s)\) on the unit circle, integrate the region between the curves \( x = f(y) = \cos s \) and \( x = g(y) = y/\tan t \) with respect to \( y = \sin s: \)

\[
\int_0^{\sin t} \left| f(y) - g(y) \right| \, dy = \int_0^t \left( \cos s - \frac{\sin s}{\tan t} \right) \cos s \, ds
= \int_0^t 1 + \cos 2s \left( \frac{1}{2} \right) \, ds - \frac{1}{\tan t} \int_0^t \sin s \cos s \, ds
= \left[ \frac{s}{2} + \frac{\sin 2s}{4} \right]_0^t - \frac{\cos t}{\sin t} \left( \frac{\sin^2 t}{2} \right) = t^2.
\]

(b) Compute the area swept out by a vector from the origin to a point \((\cosh t, \sinh t)\) on the hyperbola \( x^2 - y^2 = 1. \)

Given a point \((x, y) = (\cosh s, \sinh s)\) on the hyperbola, integrate the region between the curves \( x = f(y) = \cosh s \) and \( x = g(y) = y/\tanh t \) with respect to \( y = \sinh s: \)

\[
\int_0^{\sinh t} \left| f(y) - g(y) \right| \, dy = \int_0^t \left( \cosh s - \frac{\sinh s}{\tanh t} \right) \cosh s \, ds
= \int_0^t \cosh^2 s \, ds - \frac{1}{\tanh t} \int_0^t \sinh s \cosh s \, ds
= \left( \frac{1}{2} e^{2t} + 2t - \frac{1}{2} e^{-2t} \right) - \frac{\cosh^2 t}{\sinh t} \left( \frac{\sinh^2 t}{2} \right) = t^2.
\]
Thus, this vector also sweeps out an area relative to the $x$ axis of $t/2$, as $t$ is varied.

5. Find the volume of the region that is common to two circular pipes of unit radii that intersect at right angles.

Let the two intersecting cylinders be described by the equations $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$. If we slice the $z$ axis, the cross sections are squares of side $2x$ or $2y$, namely $2\sqrt{1-z^2}$. Hence

$$V = \int_{-1}^{1} 4(1 - z^2) \, dz = 2 \int_{0}^{1} 4(1 - z^2) \, dz = 8 \left[ z - \frac{z^3}{3} \right]_0^1 = \frac{16}{3}.$$ 

6. The region bounded by the curve $(x - R)^2 + y^2 = a^2$ is rotated about the $y$ axis, where $R > a$ to obtain a torus of minor radius $a$ and major radius $R$.

(a) Use the method of washers to find the volume of the torus.

Slice the torus in the $y$ direction, creating washers of outer radius $R + \sqrt{a^2 - y^2}$ and inner radius $R - \sqrt{a^2 - y^2}$. The area of each washer is

$$A(y) = \pi \left[ \left( R + \sqrt{a^2 - y^2} \right)^2 - \left( R - \sqrt{a^2 - y^2} \right)^2 \right] = 4\pi R \sqrt{a^2 - y^2},$$

so that the volume $V$ of the torus is given by

$$V = 4\pi R \int_{-a}^{a} \sqrt{a^2 - y^2} \, dy.$$ 

We recognize that $\int_{-a}^{a} \sqrt{a^2 - y^2} \, dy$ is just half the area of a circle of radius $a$.

$$V = 4\pi R \frac{\pi a^2}{2} = 2\pi^2 Ra^2.$$ 

(b) Use the method of shells to find the volume of the torus.

Slice the torus in the $x$ direction, creating cylindrical shells of radius $x$ and height $2\sqrt{a^2 - (x - R)^2}$.

The volume $V$ of the object is thus, on letting $u = x - R$,

$$V = 4\pi \int_{R-a}^{R+a} x \sqrt{a^2 - (x - R)^2} \, dx = 4\pi \int_{-a}^{+a} (u + R) \sqrt{a^2 - u^2} \, du = 4\pi R \frac{\pi a^2}{2} = 2\pi^2 Ra^2.$$
7. Let $f$ be a strictly increasing differentiable function on $[a, b]$ with inverse $g$. Consider the statement:

$$2\pi \int_a^b x[f(x) - f(a)] \, dx = \pi \int_{f(a)}^{f(b)} [b^2 - g^2(y)] \, dy.$$ 

(a) Establish this result with a picture for the case where $a > 0$ and $f(a) > 0$ by illustrating each integral as a volume.

Consider the volume obtained by revolving the hatched area in the diagram below about the $y$ axis. The left-hand integral slices this volume along the red vertical lines; this corresponds to the method of shells. The right-hand integral slices the very same volume along the blue horizontal lines; this corresponds to the method of cross sections. For illustration purposes we have revolved only from 0 to $4\pi/3$ radians.

(b) Perform a substitution followed by an integration by parts to provide an independent proof of this result.

On substituting $x = g(y)$ in the integral on the right-hand side, and integrating by parts, we obtain

$$\pi \int_a^b [b^2 - x^2]f'(x) \, dx = \pi [(b^2 - x^2)f(x)]_a^b + \pi \int_a^b 2xf(x) \, dx = -\pi (b^2 - a^2)f(a) + 2\pi \int_a^b xf(x) \, dx$$

$$= -2\pi f(a) \int_a^b x \, dx + 2\pi \int_a^b xf(x) \, dx = 2\pi \int_a^b x[f(x) - f(a)] \, dx.$$