Math 118: Honours Calculus II  
Winter, 2017  
Assignment 4  
March 4 due Wednesday, March 15, 2017

1. Let $f$ be a continuous function on $[a, b]$ such that $\int_a^b f(x) g(x) \, dx = 0$ for all continuous functions $g$ satisfying $\int_a^b g(x) \, dx = 0$. Prove that $f$ is constant on $[a, b]$.

Hint: recall that if $g$ is continuous on $[a, b]$ and $\int_a^b g^2(x) \, dx = 0$, then $g(x) = 0$ for all $x \in [a, b]$. Consider $g = f - c$, where $c$ is the average value of $f$ on $[a, b]$.

Let $c = \frac{1}{b-a} \int_a^b f$ be the average value of $f$ on $[a, b]$. Then

$$\int g = \int_a^b (f - c) = \int_a^b f - c(b - a) = 0$$

and

$$\int g^2 = \int (f - c)g = \int fg - c \int g = 0.$$

Since $g$ is continuous on $[a, b]$, we conclude that $g(x) = 0$ on $[a, b]$ and hence $f = c$ on $[a, b]$.

2. Find

$$\int \frac{1 - \sqrt{x + 1}}{1 + \sqrt{x + 1}} \, dx.$$ 

First let $u = x + 1$ then let $t^6 = u$.

$$\int \frac{1 - \sqrt{x + 1}}{1 + \sqrt{x + 1}} \, dx = \int \frac{1 - \sqrt{u}}{1 + \sqrt{u}} \, du = \int \frac{1 - t^3}{1 + t^2} \, 6t^5 \, dt$$

$$= 6 \int \frac{-t^8 + t^5}{t^2 + 1} \, dt = 6 \int \left( -t^6 + t^4 + t^3 - t^2 - t + 1 + \frac{t - 1}{t^2 + 1} \right) \, dt$$

$$= 6 \int \left( -t^6 + t^4 + t^3 - t^2 - t + 1 + \frac{t}{t^2 + 1} - \frac{1}{t^2 + 1} \right) \, dt$$

$$= 6 \left( -\frac{t^7}{7} + \frac{t^5}{5} + \frac{t^4}{4} - \frac{t^3}{3} - \frac{t^2}{2} + t + \frac{1}{2} \log(t^2 + 1) - \arctan t \right) + C$$

$$= -\frac{6(x + 1)^{7/6}}{7} + \frac{6(x + 1)^{5/6}}{5} + \frac{3(x + 1)^{2/3}}{2} - 2(x + 1)^{1/2} - 3(x + 1)^{1/3}$$

$$+ 6(x + 1)^{1/6} + 3 \log((x + 1)^{1/3} + 1) - 6 \arctan(x + 1)^{1/6} + C.$$
3. Find

(a) \[ \int \frac{1}{x^4 - 1} \, dx \]

\[
\frac{1}{(x^2 - 1)(x^2 + 1)} = \frac{1}{(x - 1)(x + 1)(x^2 + 1)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x^2 + 1}.
\]

Hence

\[ 1 = A(x^3 + x^2 + x + 1) + B(x^3 - x^2 + x - 1) + C(x^2 - 1), \]

from which we see that

\[ 0 = A + B, \quad 0 = A - B + C, \quad 0 = A + B, \quad 1 = A - B - C. \]

These equations have the solution \( A = \frac{1}{4}, \quad B = -\frac{1}{4}, \quad C = -\frac{1}{2}. \) Hence

\[ \int \frac{1}{x^4 - 1} \, dx = \frac{1}{4} \log |x - 1| - \frac{1}{4} \log |x + 1| - \frac{1}{2} \arctan x + C. \]

(b) \[ \int \frac{x(x^4 + x^3 - 1)}{x^4 - 1} \, dx \]

\[ \int \left( x + \frac{1}{x^4 - 1} \right) \, dx = \frac{x^2}{2} + x + \frac{1}{4} \log |x - 1| - \frac{1}{4} \log |x + 1| - \frac{1}{2} \arctan x + C. \]

(c) \[ \int \frac{x^3}{x^{16} - 1} \, dx \]

Hint: Would it perhaps help to do a substitution before applying the method of partial fractions?

Let \( u = x^4. \) Then the integral becomes

\[ \frac{1}{4} \int \frac{1}{u^4 - 1} \, du = \frac{1}{16} \log |u - 1| - \frac{1}{16} \log |u + 1| - \frac{1}{8} \arctan u + C \]

\[ = \frac{1}{16} \log |x^4 - 1| - \frac{1}{16} \log |x^4 + 1| - \frac{1}{8} \arctan x^4 + C. \]
4. (a) Find
\[ \int \frac{1}{x^3 + 1} \, dx. \]

First note that when \( n \) is odd, we may replace \( x \) by \( -x \) in the well known result \( x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \ldots + 1) \) to obtain \( x^n + 1 = (x + 1)(x^{n-1} - x^{n-2} + \ldots + 1) \).

For example, \( x^3 + 1 = (x + 1)(x^2 - x + 1) \). Therefore, we seek a partial fraction decomposition of the form

\[ \frac{1}{x^3 + 1} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 - x + 1}, \]

which requires that \( 1 = A(x^2 - x + 1) + (Bx + C)(x + 1) \).

Hence

\[
\begin{align*}
1 &= A + C, \\
0 &= -A + B + C, \\
0 &= A + B,
\end{align*}
\]

which yields \( A = 1/3 \), \( B = -1/3 \), \( C = 2/3 \). Therefore

\[
\int \frac{1}{x^3 + 1} \, dx = \frac{1}{3} \int \left( \frac{1}{x + 1} + \frac{-x + 2}{x^2 - x + 1} \right) \, dx = \frac{1}{3} \left( \log |x + 1| + \int \frac{-\frac{1}{2}(2x - 1) + \frac{3}{2}}{x^2 - x + 1} \, dx \right)
\]

\[
= \frac{1}{3} \log |x + 1| - \frac{1}{6} \log |x^2 - x + 1| + \frac{3}{2} \int \frac{1}{(x - \frac{1}{2})^2 + \frac{3}{4}} \, dx
\]

\[
= \frac{1}{3} \log |x + 1| - \frac{1}{6} \log |x^2 - x + 1| + \frac{1}{\sqrt{3}} \arctan \left( \frac{2x - 1}{\sqrt{3}} \right) + K.
\]

(b) Use part (a) to compute
\[ \int \frac{1}{x^6 + 1} \, dx. \]

Hint: Notice that \( x^6 + 1 = (x^2)^3 + 1 \) and \( (x^2)^2 - x^2 + 1 = (x^2 + 1)^2 - 3x^2 \).

Using part (a), we see that

\[ \frac{1}{x^6 + 1} = \frac{1}{(x^2)^3 + 1} = \frac{1}{3} \left( \frac{1}{x^2 + 1} + \frac{-x^2 + 2}{x^4 - x^2 + 1} \right). \]

We factorize \( x^4 - x^2 + 1 \) into its irreducible quadratic factors by completing the square in the following manner,

\[ x^4 - x^2 + 1 = (x^2 + 1)^2 - 3x^2 = \left( x^2 + 1 - \sqrt{3}x \right) \left( x^2 + 1 + \sqrt{3}x \right). \]

Then decompose

\[ \frac{-x^2 + 2}{x^4 - x^2 + 1} = \frac{Ax + B}{x^2 + \sqrt{3}x + 1} + \frac{Cx + D}{x^2 - \sqrt{3}x + 1}. \]
which yields \(-x^2 + 2 = (Ax + B)(x^2 - \sqrt{3}x + 1) + (Cx + D)(x^2 + \sqrt{3}x + 1)\), and the equations
\[
2 = B + D, \\
0 = A - \sqrt{3}B + C + \sqrt{3}D, \\
-1 = -\sqrt{3}A + B + \sqrt{3}C + D, \\
0 = A + C.
\]
The solution to these equations is \(A = -\sqrt{3}/2\), \(B = D = 1\). Thus
\[
\int \frac{1}{x^6 + 1} \, dx = \frac{1}{3} \arctan x + \frac{\sqrt{3}}{12} \log \left| \frac{x^2 + \sqrt{3}x + 1}{x^2 - \sqrt{3}x + 1} \right| + \frac{1}{6} \arctan \left( 2x + \sqrt{3} \right) + \frac{1}{6} \arctan \left( 2x - \sqrt{3} \right) + K.
\]
5. (a) Find
\[
\int \frac{1}{1 + t^4} \, dt.
\]
Hint: To factorize \(1 + t^4\) first notice that \(1 + t^4 = (1 + t^2)^2 - 2t^2\).

The latter integral can be calculated using partial fraction decomposition. The denominator \(1+t^4\) can be factorized by completing the square in the following manner,
\[
1 + t^4 = (1 + t^2)^2 - 2t^2 = \left( 1 + t^2 - \sqrt{2}t \right) \left( 1 + t^2 + \sqrt{2}t \right).
\]

On setting \(\frac{1}{1 + t^4} = \frac{At + B}{t^2 + \sqrt{2}t + 1} + \frac{Ct + D}{t^2 - \sqrt{2}t + 1}\), we find
\[
1 = (At + B)(t^2 - \sqrt{2}t + 1) + (Ct + D)(t^2 + \sqrt{2}t + 1). \quad We thus obtain the system of equations
\[
1 = B + D, \\
0 = A - \sqrt{2}B + C + \sqrt{2}D, \\
0 = -\sqrt{2}A + B + \sqrt{2}C + D, \\
0 = A + C,
\]
which has the solution $A = -C = \frac{1}{2\sqrt{2}}$, $B = D = \frac{1}{2}$. Thus

$$\int \frac{1}{1+t^4} dt = \frac{1}{2\sqrt{2}} \int \frac{t + \sqrt{2}}{t^2 + t\sqrt{2} + 1} dt - \frac{1}{2\sqrt{2}} \int \frac{t - \sqrt{2}}{t^2 - t\sqrt{2} + 1} dt$$

$$= \frac{1}{4\sqrt{2}} \int \frac{2t + \sqrt{2}}{t^2 + t\sqrt{2} + 1} dt - \frac{1}{4\sqrt{2}} \int \frac{2t - \sqrt{2}}{t^2 - t\sqrt{2} + 1} dt$$

$$+ \frac{1}{4\sqrt{2}} \int \frac{\sqrt{2}}{t^2 + t\sqrt{2} + 1} dt + \frac{1}{4\sqrt{2}} \int \frac{\sqrt{2}}{t^2 - t\sqrt{2} + 1} dt$$

$$= \frac{1}{4\sqrt{2}} \log \frac{t^2 + t\sqrt{2} + 1}{t^2 - t\sqrt{2} + 1} + \frac{1}{2\sqrt{2}} \arctan \left( \sqrt{2}t + 1 \right) + \frac{1}{2\sqrt{2}} \arctan \left( \sqrt{2}t - 1 \right) + K.$$

(b) Use integration by parts to find

$$\int \frac{4t^4}{(1+t^4)^2} dt.$$

$$\int t \frac{4t^3}{(1+t^4)^2} dt = t \frac{1}{1+t^4} + \int \frac{1}{1+t^4} dt$$

$$= t \frac{1}{1+t^4} + \frac{1}{4\sqrt{2}} \log \frac{t^2 + t\sqrt{2} + 1}{t^2 - t\sqrt{2} + 1} + \frac{1}{2\sqrt{2}} \arctan \left( \sqrt{2}t + 1 \right)$$

$$+ \frac{1}{2\sqrt{2}} \arctan \left( \sqrt{2}t - 1 \right) + K,$$

upon using part (a).

(c) Evaluate

$$\int \frac{x}{\sqrt{x^3(a-x)}} dx, \text{ where } a \neq 0.$$

On writing the integrand as $\sqrt{\frac{x}{a-x}}$ we see that we should substitute

$$\frac{x}{a-x} = t^4,$$

so that

$$x = \frac{at^4}{1+t^4} = a \left( 1 - \frac{1}{1+t^4} \right)$$

and

$$dx = \frac{4at^5}{(1+t^4)^2} dt.$$

The integral then reduces to

$$a \int \frac{4t^4}{(1+t^4)^2}$$
From part (b) we then see that

\[
\int \frac{x}{\sqrt{x^3(a-x)}} \, dx = -a \frac{\sqrt{\frac{x}{a-x}}}{1 + \frac{1}{a-x}} + a \frac{\log \left| \sqrt{\frac{x}{a-x}} + \sqrt{\frac{4x}{a-x} + 1} \right|}{4\sqrt{2}} + a \frac{\arctan \left( \sqrt{\frac{4x}{a-x} + 1} \right)}{2\sqrt{2}}
\]

\[+ \frac{a}{2\sqrt{2}} \arctan \left( \sqrt{\frac{4x}{a-x} - 1} \right) + K.
\]

6. Use one of the three Euler substitutions to compute

\[
\int \frac{dx}{x + \sqrt{x^2 + x + 1}}.
\]

The substitution \( \sqrt{x^2 + x + 1} = t + x \) is useful because then \( x^2 + x + 1 = t^2 + 2tx + x^2 \) and the \( x^2 \) terms cancel. So

\[x = \frac{t^2 - 1}{1 - 2t}
\]

and

\[dx = \frac{2t(1 - 2t) + 2(t^2 - 1)}{(1 - 2t)^2} \, dt.
\]

The integral then becomes

\[
\int \frac{2t(1 - 2t) + 2(t^2 - 1)}{[2(t^2 - 1) + t(1 - 2t)](1 - 2t)} \, dt = \int \frac{2t - 2t^2 - 2}{(t - 2)(1 - 2t)} \, dt = \int \frac{2t^2 - 2t + 2}{2t^2 - 5t + 2} \, dt
\]

\[= \int \left( 1 + \frac{3t}{(t - 2)(2t - 1)} \right) \, dt = \int \left( 1 + \frac{2}{(t - 2)} - \frac{1}{2t - 1} \right) \, dt
\]

\[= t + 2 \log |t - 2| - \frac{1}{2} \log |2t - 1| + C.
\]

7. Let \( I_n = \int_0^{\pi/2} \sin^n x \, dx \) for \( n = 0, 1, 2, \ldots \)

(a) For \( n \geq 2 \), prove that

\[I_n = \left( \frac{n - 1}{n} \right) I_{n-2}.
\]

From the reduction formula for \( \sin^n x \), we find

\[
I_n = \int_0^{\pi/2} \sin^n x \, dx = \left[ \frac{-1}{n} \sin^{(n-1)} x \cos x \right]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \sin^{(n-2)} x \, dx = \frac{n-1}{n} I_{n-2}.
\]
(b) For \( n \in \mathbb{N} \), prove that
\[
I_{2n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1) \pi}{2 \cdot 4 \cdot 6 \cdots 2n} > 0.
\]
Since \( I_0 = \int_{0}^{\pi/2} 1 \, dx = \pi/2 \), we see from part (a) that \( I_2 = \pi/4 \).
Suppose that the formula holds for a particular value of \( n \). Then
\[
I_{2n+2} = \left( \frac{2n + 1}{2n + 2} \right) I_{2n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1) \cdot [2(n + 1) - 1] \pi}{2 \cdot 4 \cdot 6 \cdots 2n \cdot 2(n + 1))}.
\]
By induction, we see that the formula holds for all \( n \in \mathbb{N} \); hence \( I_{2n} > 0 \) for all \( n \in \mathbb{N} \).

(c) For \( n \in \mathbb{N} \), prove that
\[
I_{2n+1} = \frac{2 \cdot 4 \cdot 6 \cdots 2n}{3 \cdot 5 \cdot 7 \cdots (2n + 1)}.
\]
Since \( I_1 = \int_{0}^{\pi/2} \sin x \, dx = \left[ -\cos x \right]_{0}^{\pi/2} = 1 \), we see from part (a) that \( I_3 = 2/3 \).
Suppose that the formula holds for a particular value of \( n \). Then
\[
I_{2n+3} = \left( \frac{2n + 2}{2n + 3} \right) I_{2n+1} = \frac{2 \cdot 4 \cdot 6 \cdots 2n \cdot 2(n + 1)}{3 \cdot 5 \cdot 7 \cdots (2n + 1) \cdot [2(n + 1) + 1]}.
\]
By induction, we see that the formula holds for all \( n \in \mathbb{N} \).

(d) Show that \( I_{2n+2} \leq I_{2n+1} \leq I_{2n} \) for all \( n \in \mathbb{N} \).
Since \( 0 \leq \sin x \leq 1 \) for \( x \in [0, \pi/2] \), upon multiplication by \( \sin^{2n} x \) and \( \sin^{2n+1} x \), respectively, we see that
\[
\sin^{2n+2} x \leq \sin^{2n+1} x \leq \sin^{2n} x.
\]
From Theorem 5.5.1, we then deduce that \( I_{2n+2} \leq I_{2n+1} \leq I_{2n} \).

(e) Use part (d) to show that
\[
\frac{2n + 1}{2n + 2} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1
\]
and deduce
\[
\lim_{n \to \infty} \frac{I_{2n+1}}{I_{2n}} = 1.
\]
This follows directly upon division of the inequality in part (c) by \( I_{2n} \), since
\[
\frac{I_{2n+2}}{I_{2n}} = \frac{2n + 1}{2n + 2}.
\]
Since
\[
\lim_{n \to \infty} \frac{2n + 1}{2n + 2} = 1,
\]
we see from the Squeeze Principle that \( \lim_{n \to \infty} \frac{I_{2n+1}}{I_{2n}} = 1 \).
(f) Prove that

\[
\lim_{n \to \infty} \left( \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \cdots \cdot \frac{2n}{2n - 1} \cdot \frac{2n}{2n + 1} \right) = \frac{\pi}{2}.
\]

This formula is usually written as the infinite Wallis product

\[
\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{(2n)^2}{4n^2 - 1} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n - 1)(2n + 1)} = \left( \frac{2 \cdot 2}{1 \cdot 3} \right) \left( \frac{4 \cdot 4}{3 \cdot 5} \right) \left( \frac{6 \cdot 6}{5 \cdot 7} \right) \cdots;
\]

however, this is not a very efficient way of calculating \( \pi \).

Inserting the expressions in part (b) and (c) into the limit in part (e), we obtain

\[
1 = \lim_{n \to \infty} \left( \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdots (2n - 1)} \right) \left( \frac{2}{\pi} \right) \left( \frac{2 \cdot 4 \cdot 6 \cdots 2n}{3 \cdot 5 \cdot 7 \cdots (2n + 1)} \right),
\]

from which the result follows.